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MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC
LOS ANGELES 1971

A meeting of the Association for Symbolic Logic was held on March 25 and 26, 1971 at the Beverly Hilton Hotel, Beverly Hills, California in conjunction with the annual meeting of the American Philosophical Association, Pacific Division. The Council of the Association met at dinner on Thursday.

Invited addresses were delivered by Professor Saharon Shelah on *Any two elementarily equivalent models have isomorphic ultrapowers* and by Professor Dana S. Scott on *A model theory for the λ-calculus.* Professors Alfred Tarski and Yiannis Moschovakis presided over the two invited addresses. On Friday an invited symposium on the topic, Rudolf Carnap, was held. This symposium was cosponsored by the American Philosophical Association. The speakers were Professors Carl G. Hempel, Jaakko Hintikka, and Richard C. Jeffrey. Professor Maria Reichenbach presided over the symposium. The first twenty papers below were presented in person, Professors Daniel Gallin, Jon Barwise, Herbert Enderton, Carl Gordon, Donald Potts, Dana Scott, and I. Reznikoff presiding. The last six papers below were submitted by title.

Professor Richard Montague served as Chairman of the Program Committee. The program was completed before his death on March 7, 1971.

DAVID KAPLAN

WELLS, GARDNER S. *A calculus of contexts.*

Many fields of study (e.g., sociology) need a method of interrelating noncoextensive contexts. This calculus is sketched here, heuristically, as an approach to the problem. Because of its extra-logical applications, symbols are limited to the standard keyboard.

Assume a class V with subclasses x, y, z, w, x_j, etc., for which $\cap$, $\cup$ and $'$ indicate class products, sums and complements. Let a, b, a_j, etc., represent ordered pairs of subclasses. Using $\leftrightarrow$ for "represents", and letting $a \leftrightarrow x;y$, $a_j \leftrightarrow x_j;y_j$, and $b \leftrightarrow z;w$, various operations are defined as follows:

$-a \leftrightarrow x';y'$, $\#a \leftrightarrow y;x$ and $@a \leftrightarrow y';x'$.

$\&(a_1 \cdots a_n) \leftrightarrow (x_1 \cap \cdots \cap x_n);(y_1 \cup \cdots \cup y_n)$ and

$\underline{\&}(a_1 \cdots a_n) \leftrightarrow (x_1 \cap \cdots \cap x_n);(y_1 \cap \cdots \cap y_n)$.

$\$(a_1 \cdots a_n) \leftrightarrow -\&(-a_1 \cdots -a_n)$ and $\underline{\$}(a_1 \cdots a_n) \leftrightarrow -\underline{\&}(-a_1 \cdots -a_n)$.

$o \leftrightarrow \&(a - a)$, $u \leftrightarrow -o$, $\underline{o} \leftrightarrow \underline{\&}(a - a)$ and $\underline{u} \leftrightarrow -\underline{o}$.

$S\ a::b \leftrightarrow x = z$ and $y = w$.

It is easily shown that:

(1) If each of P, Q and R is replaced by one symbol for complementation, no two the same, S a::PPa, S PQa::QPa and S Pa::QRa.

(2) $(-, \&, o)$ and $(-, \underline{\&}, \underline{o})$ are distinct and complete Boolean algebras.

(3) In defining \$, # may replace $-$; the same holds for $\underline{\$}$ and @.

Further definitions of interest are:

$a/b \leftrightarrow \&(\underline{\&}(au)\underline{\$}(bu))$ $\quad$ $(a/b \leftrightarrow x;w)$

$Ct'a \leftrightarrow \$(a \# a)/o$ $\quad$ $((x \cup y);V$, termed the *context* of a).

In a forthcoming article this calculus will be presented as an independent system, requiring only two primitive operators.

LEBLANC, HUGUES. *Truth-value semantics for the modal logics* QM, QS4, *and* QS5.

Σ being a set of functions from the atomic wffs of QM (von Wright's *M* with quantifiers) to $\{T, F\}$, α being a member of Σ, and *R* being a reflexive relation on *R*, take a wff *A* of QM to be true on the triple $\langle \Sigma, \alpha, R \rangle$ if:

(i) in case *A* is atomic $\alpha(A = T$

(ii) in case A is of the sort $\sim B$, B is not true on $\langle \Sigma, \alpha, R \rangle$,

(iii) in case A is of the sort $B \supset C$, B is not true on $\langle \Sigma, \alpha, R \rangle$ or C is,

(iv) in case A is of the sort $(\forall X)B$, the result $B(P/X)$ of putting P for X in B is true on $\langle \Sigma, \alpha, R \rangle$ for every individual parameter P of QM, and

(v) in case A is of the sort $\Box B$, B is true on $\langle \Sigma, \alpha', R \rangle$ for every member α' of Σ such that $R(\alpha, \alpha')$.

It can be shown that a wff A of QM is provable in QM if and only if A is true on every triple $\langle \Sigma, \alpha, R \rangle$ of the sort just described. And like results obtain for QS4 (S4 with quantifiers) when R is required to be transitive as well as reflexive, and for QS5 (S5 with quantifiers) when R is required to be transitive and symmetrical as well as reflexive.

It is assumed here that the Barcan formula (provable in QS5) counts as an axiom of QM and QS4. If a strong completeness proof for QM, QS4, and QS5 is to be had, Σ must be construed as a sequence of indexed functions, and R as a relation on the indices of these assignments.

SUGAR, ALVIN C. *A logical requiem for relativity.*

This paper is concerned with the greatest scandal in the history of science. The theory of relativity can be shown to be counter factual by an almost childish example. Let me, by way of interjection, refer to a very appropriate legend. Procrustes was a celebrated legendary highwayman of Attica who tied his victims upon an iron bed and, as the case required, either stretched or cut off their legs to adapt them to its length. A Procrustean bed refers therefore to a theory to which facts are arbitrarily adjusted. *Relativity is a Procrustean bed.* Instead of fitting the theory to the facts, the facts are fitted to the theory. I call for the substantial application of logic and axiomatic procedures to physics. How can the physicists dare to construct theories without the essential and modern tools required for their solid fabrication. The failure of relativity as a physical theory in turn collapses its parent theory, Maxwell's electromagnetism, and this in turn collapses another offspring of electromagnetism, namely, quantum dynamics. To continue with my iconoclastic destruction, let me add that I reject the Michelson-Morley experiment for it was born in bias and enshrined in contradiction. This extensive annihilation of large portions of modern physics creates a vacuum into which we propose to erect my *generalized unified field theory* developed within the framework of *strict axiomatization.*

We alter Newton's law of universal gravitation by adding two correction terms. These terms have the effect of accounting for (1) the advance of perihelia in quasi-elliptical orbital motion and (2) atomic repulsion. We formulate a modified Gauss-Bush invariant mass, variant charge foundation of electrodynamics, which unlike Maxwell's electromagnetism is compatible with Newtonian dynamics. We give a more logical formulation of the molecular and the kinetic theories of matter in terms of an explicit quantitative formulation of atomic repulsion. We properly reduce my axiomatic formulation of thermodynamics to the kinetic theory of matter. Of the many objections I have to relativity, I have elected to select the following as a crucial defect and concentrate on it. When the points of light A and B move in opposite directions from a source S, A to the left and B to the right, we must conclude, using the simplest accepted laboratory techniques, that the rate of separation of these points is $2c$. This is *inviolate*—this is fact. For that matter, to deny that this is fact is to deny the validity of any or all empirical procedures and hence the rationality of man. It is sheer insanity, then, for anyone to present us with a theory that contradicts this basic empirical fact, a theory which requires that this velocity be c.

SELDIN, JONATHAN P. *The paradox of Kleene and Rosser.*

In their [IFL], Kleene and Rosser showed that the Richard Paradox can be set up in certain systems of illative combinatory logic, and in his [PKR] Curry studied this paradox in detail for a system with stronger postulates. In this paper it is shown that the paradox can be derived from weaker postulates.

The most important of these postulates can be stated, using the notation of [CLg. II, §12B4] and [SIC, §2A3], as follows: if M is a sequence of terms and if x is a variable which does not occur (free) in M, X, or Y, then

$$(1) \qquad M, Xx \vdash Yx \;\&\; \mathrm{Can}_1(X) \rightarrow M \vdash Xx \supset_x Yx.$$

Then the postulate of Kleene and Rosser [IFL] can be obtained by specifying that $\text{Can}_1\ (X)$ holds just when there is a term U such that $M \vdash XU$ (so that the terms X such that $\text{Can}_1\ (X)$ depends on M), and the postulate of Curry [PKR] is that obtained by assuming that $\text{Can}_1\ (X)$ holds for all terms X.

In this paper, it is shown that if we begin with assumptions about Can_1 satisfied by the system $\mathscr{F}_{21}$ of [CLg. II, §15B] (which can be proved to be consistent if the canonical terms are taken to be the canobs of [CLg. II, §12B3]), and if we then assume in addition that there is a term T such that $\text{Can}_1\ (T)$ and

$$M \vdash TX \rightleftarrows M \vdash X,$$

then we can, using (1), derive the paradox.

REFERENCES. [PKR] CURRY, H. B., *The paradox of Kleene and Rosser*, ***Transactions of the American Mathematical Society***, vol. 50 (1941), pp. 454–516.

[CLg. II] CURRY, H. B., HINDLEY, J. R., and SELDIN, J. P., ***Combinatory Logic***, vol. 2. (To be published in Amsterdam by North-Holland, probably in 1971).

[IFL] KLEENE, S. C. and ROSSER, J. B., *The inconsistency of certain formal logics*, ***Annals of Mathematics***, (2) vol. 36 (1935), pp. 630–636.

[SIC] SELDIN, J. P., *Studies in illative combinatory logic*, Dissertation, Amsterdam, 1968.

MOSIER, RICHARD D. *Recursive functions and the tensor calculus.*

A "primitive" recursive function such as $Q(x) = x'$ is read "the function of x is its successor"; but of course we have no way of knowing whether the "successor" in question is $x + 1$, $x + 2, x + 3, \cdots, x + n$.

What is needed is a way of assigning particular values to the "successors" of the function without impairing the generality of the function. For this purpose, we can use *indices* of the function, for example,

$$Q(x)_{ik} = A'_{ik} \qquad (i, k = 1, 2, 3)$$

which indicates that we are dealing with a second-order recursive function in which there are as many "successors" of the function as there are "components" in the corresponding tensor indices.

Thus A'_{ik} is the "successor" of $Q(x)_{ik}$, which in matrix form displays its "components" in the following way:

$$\left\| A'_{ik} \right\| = \left\| \begin{matrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{matrix} \right\|$$

The matrix form of A'_{ik} indicates that the "successors" of a recursive *function* have been transformed into the "components" of a recursive *relation*. But since the "components" of A'_{ik} have been displayed in matrix form, perhaps, it is also possible to display the "successors" of $Q(x)_{ik}$ in matrix form:

$$Q(x)_{ik} = \begin{bmatrix} Q(x_{11} & x_{12} & x_{13}) \\ Q(x_{21} & x_{22} & x_{23}) \\ Q(x_{31} & x_{32} & x_{33}) \end{bmatrix}$$

Consequently, we note that (substituting l and m for i and k) the relation between $Q(x)_{ik}$ and A'_{ik} can be expressed in the following formulas:

$$Q(x'_i) = a_{i'l}x_l \qquad Q(x'_k) = a_{k'm}x_m$$
$$Q(x_l) = a_{i'l}x'_i \qquad Q(x_m) = a_{k'm}x'_k.$$

By appropriate transpositions and substitutions, the formulas expressed above can be reduced to:

$$A'_{ik} = a_{i'l}a_{k'm}Q(x)_{lm}.$$

We observe in conclusion that the transformation of the "successors" of a recursive *function* into the "components" of a recursive *relation* is the *logical* equivalent of a change of coordinate systems, but the *mutual* (dialectical) recursiveness of the systems (formulas) permits us to express the *evolution* of the systems of "successors" and "components" as a recursive equilibration of the *process* of recursion, that is, as a *recursive* logic.

SINGLETARY, W. E. *Representation of many-one degrees by partial propositional calculi.*
M. D. Gladstone (Trans. Amer. Math. Soc., vol. 118 (1965), pp. 192–210), has shown that every recursively enumerable degree of unsolvability can be represented by a partial implicational propositional calculus (p.i.p.c.). Since it is a well-known result that not every r.e. many-one degree can be represented by a first order theory the question as to what restricted degrees can be represented by p.i.p.c. seems a natural. We have obtained the following rather surprising result.

THEOREM. *Given any arbitrary r.e. many-one degree d one can effectively construct a p.i.p.c. with decision problem of degree d.*
The proof utilizes a recent result (C. E. Hughes, Ross Overbeek and W. E. Singletary, ***Bulletin of the American Mathematical Society***, vol. 77 (1971), p.p. 462–472.) that every r.e. many-one degree can be represented by a semi-Thue system. Given this result the construction and proof follow rather closely those given by us in showing that every r.e. degree can be represented by a p.i.p.c. (***Journal of the Faculty of Science,*** University of Tokyo, XIV (1967), pp. 25–58).

HUGHES, CHARLES E. *Representation of many-one degrees by Markov algorithms.*
Markov algorithms, which were first defined by A. A. Markov in the early 1950's, have been extensively studied by both logicians and computer scientists, e.g., Mendelson [***Introduction to mathematical logic,*** Van Nostrand Co., Princeton, 1966] and Galler and Perlis [***A view of programming languages,*** Addison Wesley, Reading, Mass., 1970]. In connection with these systems a number of interesting questions arise as to the structure of the various general decision problems associated with them. In particular, we have investigated the degree representations of the general word, halting and confluence problems and have effectively shown that every r.e. many-one degree of unsolvability may be represented by each of these. The technique used to achieve this result is to demonstrate an effective procedure which, when applied to an arbitrary Turing machine T, produces a Markov algorithm whose word, halting and confluence problems are of the same many-one degrees as the derivability, halting and confluence problems for T, respectively. This, combined with the results of Overbeek [see the next abstract], gives us the desired results. Moreover, we have shown this to be best possible in the sense that every r.e. one-one degree of unsolvability may not be represented by any of these general decision problems. Finally, as a direct corollary to this, we have that the class of Markov algorithms is computationally equivalent to the class of total recursive functions, in that every total recursive function is computable by a Markov algorithm which always halts.

OVERBEEK, ROSS. *Representation of many-one degrees by the word problem for Thue systems.*
Recent results (C. E. Hughes, Ross Overbeek and W. E. Singletary, ***Bulletin of the American Mathematical Society,*** vol. 77 (1971), pp. 467–472.) have shown methods of representing any recursively enumerable many-one degree by either the decision problems (halting, derivability, and confluence) of Turing machines or the word problem for semi-Thue systems. One naturally wonders whether the degree could also be represented by word problems of Thue systems. We have shown the following result.

THEOREM. *Given an arbitrary r.e. many-one degree d one can effectively construct a Thue system whose word problem is of degree d.*
The proof involves the construction of a Turing machine M whose confluence problem is of degree d. A Thue system T is then constructed which simulates the operations of the Turing machine closely enough to allow one to establish that the confluence problem of M and the word problem of T are many-one equivalent.

SMITH, PERRY. *Some special cases of Montague's recursion theory.*
The standard analytic hierarchy of relations among numbers and infinite sequences is

obtained by considering the definability of such relations in the structure with universe $\omega \cup \omega^{\omega}$ and basic relations zero, successor, and function value, using the language of finite type theory with all variables except individual variables ranging over hereditarily finite sets. A second characterization is obtained by using countable sets instead of finite sets.

A recursion theory over the ordinals less than a given infinite cardinal $\mathfrak{m}$ is obtained, in which the only basic relation is the one holding between an ordinal and the set of all smaller ordinals, and the variables of higher type range over sets hereditarily of power $< \mathfrak{m}$.

SOLON, T. P. M. *Composition and quantification.*

Virtually all logicians agree that compositional arguments are not formally fallacious. (A) Most writers prefer to list such arguments among the informal fallacies of ambiguity. (B) Some even go so far as to deny that compositional inferences contain any error in reasoning whatsoever.

My own view of the matter is that the advocates of (A) and (B) are mistaken. Consider the following typical example of composition:

Every living thing has a mother. Hence there is some individual which is the mother of every living thing.

In terms of quantification this translates into:

1. $$(x)[Lx \rightarrow (Ey)Myx] / \therefore\ (Ey)[(x)Lx \rightarrow Myx].$$

This sort of argumentation is obviously formally invalid. Specifically it involves an illicit interchange in the scope of the universal and existential quantifiers. Since all instances of composition exhibit such a structure, they are formally fallacious, and so positions (A) and (B) must be abandoned.

WOODRUFF, PETER W. *A new approach to possible objects.*

The standard approach to possible objects in contemporary modal logic is, in my opinion, open to a number of philosophical objections. We present a new semantics based on the principle that a simple property is "true of" a nonexistent object just in case it is true of that object in all worlds in which the latter exists. This semantics can be shown to be consistent and complete with respect to an appropriate deductive system. An interesting feature of the system is that it provides a fruitful application for three-valued logic.

GALLIN, DANIEL. *Systems of intensional logic.*

Montague's system *IL* (intensional logic) is a synthesis of Church's theory of types with modal logic, capable of treating such troublesome grammatical entities as intensional verbs, adjectives and prepositions. Let e, t, s be distinct entities; the set T of *types* is the smallest set such that (i) $e, t \in T$; (ii) if $\alpha, \beta \in T$ then $\langle \alpha, \beta \rangle, \langle s, \alpha \rangle \in T$. *Terms of type* α are characterized as follows: (i) variables or constants of type α (denumerably many) are terms of type α; (ii) if A, B, C, D are terms of types $\langle \alpha, \beta \rangle, \alpha, \alpha, \langle s, \alpha \rangle$ respectively, and v is a variable of type γ, then $[AB]$, $\lambda v\, B$, $[B \equiv C]$, $^\wedge B$, $^\vee D$ are terms of types $\beta, \langle \gamma, \alpha \rangle, t, \langle s, \alpha \rangle, \alpha$ respectively. A *model based on* nonempty sets D and I is a system $M = \langle (M_\alpha)_{\alpha \in T}, m \rangle$ such that $M_e = D$, $M_t = \{0, 1\}$, $M_{\langle \alpha, \beta \rangle} = M_\beta^{M_\alpha}$, $M_{\langle s, \alpha \rangle} = M_\alpha^I$, and $m(c)(i) \in M_\alpha$ when c is a constant of type α and $i \in I$. Let J consist of all *assignments* over M; i.e., functions φ mapping variables of type α into M_α for all $\alpha \in T$. Given $i \in I$, $\varphi \in J$ we define, for each term A of type α, a *value* $V_{i,\varphi}(A) \in M_\alpha$. The clauses are the usual ones, together with: $V_{i,\varphi}(^\wedge B)(j) = V_{j,\varphi}(B)$ and $V_{i,\varphi}(^\vee D) = V_{i,\varphi}(D)(i)$. A *formula*, or term of type t, will always have value 0 or 1, and the notions of semantical consequence, etc., are as usual. The sentential connectives, quantifiers and modal operators can all be defined in *IL*.

A Henkin-type completeness theorem is proved for *IL*, using generalized models. Several alternative formulations of higher-order modal logic are described and compared with *IL*; in one of these systems a natural prenex form theorem obtains.

POWELL, WILLIAM C. *An axiomatization of set theory with predication as a relation.*

We consider another axiomatization of set theory. It is a first-order theory with equality, the membership relation, a new binary relation called prediction, and a constant V. Sets are defined to be elements of V. Classes are defined to be collections of sets. The variables P, Q are defined to range over classes. Thus, $\forall P\Phi(P)$ is short for

$$\forall x(\forall y(y \in x \to y \in V) \to \Phi(x)).$$

Predication is denoted by juxtaposition, and we only consider classes on the left of predication. The axioms are

(A) $x \in y \in V \to x \in V,$

(B) $\forall x \in V(Px \leftrightarrow x \in P),$

(C) $\forall x \in V(Px \leftrightarrow Qx) \to P = Q,$

(D) $\forall \vec{x} \in V\exists Q\forall y(Qy \leftrightarrow \Phi(\vec{P}; \vec{x}, y))$

where Φ is a formula such that (i) all the free variables are displayed, (ii) the P's are the only variables occurring on the left in predication, (iii) all the P's occur only on the left in predication, and (iv) V does not occur.

Except for regularity, all the axioms of Zermelo-Fraenkel set theory are derivable in the theory. Also the existence of indescribable and ineffable cardinals is derivable. If the theory is consistent, then the theory plus $\mathbf{V} = \mathbf{L}$ is consistent. The consistency of the theory can be established assuming the existence of a 2-valued measurable cardinal. Moreover, the theory can be shown to be consistent from assumptions consistent with $\mathbf{V} = \mathbf{L}$. Models of the theory are closely related to Kunen's notion of M-ultrafilter.

OLLMANN, L. TAYLOR. *Operators preserving elementary equivalence.*

Certain operators on relational structures (such as definable homomorphisms, direct unions, reduced products, limit ultrapowers and the generalized products of Feferman and Vaught (***Fundamenta Mathematicae***, vol. 47)) all preserve elementary equivalence. That is to say the first order theories of the structures to which the operator is applied determine the first order theory of the image structure.

A more general class of such operators preserving elementary equivalence is defined and a subclass preserving elementary extensions is isolated.

The technique is to define a topology-like structure on the class of relational structures. The operators are then defined to be those functions of relational structures with certain "continuity" properties. The proof that these operators preserve elementary equivalence uses a game theoretic characterization of elementary equivalence introduced by A. Ehrenfeucht.

Structure theorems are obtained which make the operators relatively easy to construct and work with. They are closed under composition and frequently preserve equivalence with respect to stronger languages. In fact they are readily altered to preserve equivalence in infinity languages.

GEISER, JAMES R. *A formalization of Esenin-Volpin's proof theory with the aid of nonstandard analysis.*

In 1959 Esenin-Volpin presented to the Warsaw Symposium on the Foundations of Mathematics a paper sketching a proof of the consistency of Zermelo-Fraenkel set theory (ZF). Intuitively the idea was that very large sets among the heriditarily finite sets (HF) could be used to instantiate the axiom of infinity, while the other axioms of ZF are modeled in HF as usual. The distinction between small (or feasible) sets and very large sets can be partially formalized in nonstandard analysis using finite sets versus pseudo finite sets. We proceed as follows. A proof theory $\mathscr{G}_n$ is developed for the hereditarily finite sets over a set of n urelements along the lines of Fitch including a Carnap's rule: $\{A(t) \mid t \text{ any closed term}\} \vdash \forall xA(x)$. After extending these constructions to a nonstandard integer n_0 a certain subcollection $\mathscr{G}_{EV} \subset \mathscr{G}_{n_0}$ is chosen to represent Esenin-Volpin's proof theory. Roughly speaking, a subset of the constant terms is singled out to act as the "feasible" terms. A proof tree T of $\mathscr{G}_{n_0}$ is in $\mathscr{G}_{EV}$ iff only feasible terms occur in the subtree $\breve{T}$ (of T) in which the Carnap's rule has been restricted to $\{A(t) \mid t \text{ feasible}\} \vdash \forall xA(x)$. (Note that terms may arise in the course of proving existential sentences in $\breve{T}$). By means of these ideas a nonclassical proof theory $\vdash_{\mathscr{g}}$ is developed. $\vdash_{\mathscr{g}}$ is shown to be consistent and closed under modus ponens as well as other derived rules, e.g. $\vdash_{\mathscr{g}} A \vee B \Leftrightarrow \vdash_{\mathscr{g}} A$ or $\vdash_{\mathscr{g}} B$, $\vdash_{\mathscr{g}} \exists xA(x) \Leftrightarrow \vdash_{\mathscr{g}} A(t)$, t feasible, $\vdash_{\mathscr{g}} \neg\neg A \Leftrightarrow A$. The law of the excluded middle fails in general. There are also $\vdash_{\mathscr{g}}$ proofs of the axioms of Pairing, Infinite Union, Powerset, Infinity and forms of Comprehension and Replacement. All Σ_1^1 true sentences of arithmetic are $\vdash_{\mathscr{G}}$ provable while there are π_1^1 $\vdash_{\mathscr{G}}$-undecidable sentences.

GRANT, JOHN. *Recognizable algebras of formulas.*

L is a first-order language with equality and $L_{\mathfrak{A}}$ is the diagram language for the structure $\mathfrak{A}$. Let Γ be a set of formulas of $L_{\mathfrak{A}}$. Then Γ is called a recognizable set of formulas if:

(1) the free variables in each $\varphi \in \Gamma$ are identical,

(2) there is a test formula, $T(\varphi)$, of L such that for a formula φ (with the proper free variables) $\varphi \in \Gamma$ iff $\models_{\mathfrak{A}} T(\varphi)$.

Consider such a Γ as the domain of an algebra $\hat{\mathbf{R}}$. If

(3) each algebraic operation of $\hat{\mathbf{R}}$ is expressible uniformly in the language L, and

(4) the equivalence relation $\sim$, $\varphi \sim \psi$ iff $\models_{\mathfrak{A}} \varphi \leftrightarrow \psi$, is a congruence relation in the algebra $\hat{\mathbf{R}}$, then the quotient algebra $\mathbf{R} = \hat{\mathbf{R}}/\sim$ is called a recognizable algebra of formulas.

The definition of a recognizable algebra is given in L. Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for L. $\mathbf{R}_{\mathfrak{A}}$ and $\mathbf{R}_{\mathfrak{B}}$ are called corresponding recognizable algebras if their definitions are identical in L.

THEOREM. $\mathfrak{A} \equiv \mathfrak{B}$ *iff for each pair of corresponding recognizable algebras* $\mathbf{R}_{\mathfrak{A}}$ *and* $\mathbf{R}_{\mathfrak{B}}$, $\mathbf{R}_{\mathfrak{A}} \equiv \mathbf{R}_{\mathfrak{B}}$.

COROLLARY. $\mathfrak{A} \equiv \mathfrak{B}$ *iff each pair of corresponding recognizable algebras are equationally equivalent.*

The theorem and the corollary can be extended to $L_{\alpha\alpha}$.

Let A be an algebra and θ a congruence relation on A. θ is called a recognizable congruence relation if it is defined by a formula $T(x, y)$ of L.

THEOREM. $\mathbf{A} \equiv \mathbf{B}$ *iff for every recognizable congruence relation* θ, $\mathbf{A}/\theta \equiv \mathbf{B}/\theta$.

This theorem can be extended to $L_{\alpha\beta}$.

PARSONS, CHARLES. *On a number-theoretic choice schema.* II.

As in [1], let Z_0 be elementary number theory with all elementary functions and only quantifier-free induction. We consider the results of adding axiom schemata or rules to Z_0. Let FAC be the schema

$$\forall x < a \exists y Axy \supset \exists c \forall x < a A(x, c_x)$$

(c ranges over sequence numbers). Let IR and IA be the rule and axiom schema of induction respectively. For any schema S, let $S_n^\Sigma(S_n^\Pi)$ be S restricted to $\Sigma_n(\Pi_n)$ formulae of Z_0.

In [2], IR_{n+1}^Σ is proved closed under IR_{n+2}^Π. By applying the same relativization technique to the proof of Theorem 2 of [1], we show that $IR_{n+1}^\Sigma + FAC_n^\Pi$ is also closed under IR_{n+2}^Π. It follows that IA_{n+1}^Σ is properly stronger than $IR_{n+1}^\Sigma + FAC_n^\Pi$, since IA_{n+1}^Σ can easily be seen not to be closed under IR_{n+2}^Π. The consistency of FAC_n^Π can be proved in IR_{n+1}^Σ.

Combining this work with that of [1] and [2], we have properly between IA_n^Σ and IA_{n+1}^Σ two incomparable systems, IR_{n+1}^Σ and FAC_n^Π, whose l.u.b. is still properly weaker than IA_{n+1}^Σ. The other systems considered in [1] and [2] reduce to these.

REFERENCES. [1] C. PARSONS, *On a number-theoretic choice schema and its relation to induction.* A Kino, J. Myhill, and R. E. Vesley (eds.), ***Intuitionism and proof theory***, Amsterdam, 1970, pp. 459–473.

[2] ———, *On n-quantifier induction* (to appear in this JOURNAL).

SMORYNSKI, C. *The undecidability of some intuitionistic theories of equality and order.*

Let T be an intuitionistic theory and let M_1 be the intuitionistic monadic predicate calculus on one predicate letter. For each formula A of M_1, define A to be valid in T iff A' is a theorem of T for every instance, A', of A in the language of T.

The (obvious) completeness problem is to prove: A is a theorem of M_1 iff A is valid in T. Since the provability of A implies its validity, the problem is reduced to proving: If A is not a theorem of M_1, then some instance, A', of A is not a theorem of T. The natural effective completeness problem is thus: For each formula A of M_1, an instance, A', of A must be effectively found, such that, if A is not a theorem of M_1, then A' is not a theorem of T.

By the undecidability of M_1 (Maslov, Mints, and Orevkov), an effective completeness theorem will yield the hereditary undecidability of the theory T—hereditary, since the completeness theorem holds for all subtheories of T.

Effective completeness theorems are obtained for several intuitionistic theories of equality and order, including:

(1) The theories of equality and normal equality on infinite domains. This is a minor improvement on Lifshits.

(2) The theory of an apartness relation, as described in Heyting, p. 49. (This result was obtained jointly by R. Statman and myself.)

(3) The induction-free theory of successor, given by the axioms:

$$0 \neq x',$$
$$x' = y' \supset x = y,$$
$$x \neq x'^{\cdots\prime},$$
$$x \neq 0 \supset \exists y(x = y').$$

(The addition of induction or, equivalently, a decidable equality yields a decidable theory, as shown by Lopez-Escobar.)

(4) The theory of dense linear order, obtained by adding the following to Scott's axioms for linear order (I, p. 195):

$$\exists y(x < y),$$
$$\exists y(y < x),$$
$$\exists z(x < y \supset x < z < y).$$

It follows that Scott's theory of linear order is undecidable. This settles his question (II, p. 237).

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DE JONGH, D. H. J. *Disjunction and existence under implication in intuitionistic arithmetic.*

By formalizing Kleene's notion $\Gamma \mid A$ and the argument of *Disjunction and existence under implication in elementary intuitionistic formalisms*, this JOURNAL, vol. 27 (1962), pp. 11–18, an extension is obtained of the results in that paper to formulas of the form $A \to B \vee C$ and $A \to \exists x B(x)$ with free variables. For each pair of formulas E, A of Heyting's arithmetic a formula $E \mid A$ is defined with exactly the free variables occurring in E or A. It is then provable that, if $\vdash E \to A$, then $E \mid E \vdash E \mid A$. As a corollary it follows that, if $\vdash C \to \exists x A(x)$, with x not free in C, then $C \mid C \vdash \exists x(C \to A(x))$ and, for example, since always $\vdash \neg C \mid \neg C$, if $\vdash \neg C \to \exists x A(x)$, then $\vdash \exists x(\neg C \to Ax)$. In fact, it can be shown that, whenever C fulfills Harrop's condition (*Concerning formulas of the types $A \to B \vee C$, $A \to (Ex)B(x)$ in intuitionistic formal systems*, this JOURNAL, vol. 25 (1960), pp. 27–32) of not containing "relevant" occurrences of $\vee$ and $\exists$, then, if $\vdash C \to \exists x A(x)$, also $\vdash \exists x(C \to Ax)$.

By means of a second slightly more complicated formalization a constructive proof is obtained of the following assertion.

If $f(B)$ is a propositional formula with only the propositional variable B and $f(B)$ is not provable in the intuitionistic propositional calculus, and, if furthermore A is a closed formula of Heyting's arithmetic, then $\vdash f(A)$ implies $\vdash \neg\neg A$ or $\vdash \neg\neg A \to A$.

SCHOTT, HERMANN F. *Subject and predicate calculi.*

A universe of discourse is considered in which atomic sentences have the form $f_i a_j$. (Object language symbols with numerical subscripts are designated by syntactical symbols of the same form but with literal or no subscripts. Logical symbols including concatenation are used autonomously.) The a_i are elementary subjects designating things; the f_i are elementary predicates designating attributes. The variables g_i and b_i range over attributes and things, respectively. Symbols of the forms α_i and x_i are used respectively for classes and bundles.

The class calculus arises from the axioms and rules of the propositional calculus (PC) together with those of the predicate calculus and the following definitional axioms:

(C1) $\quad .a \in \lambda bP \equiv a/bP,$

(C2) $\quad .a \in f \equiv fa,$

(C3) $\quad .\alpha_i \subset \alpha_j \equiv \forall b.b \in \alpha_i \supset b \in \alpha_j,$

(C4) $\quad .\alpha_i \simeq \alpha_j \equiv .\alpha_i \subset \alpha_j \;\&\; \alpha_j \subset \alpha_i,$

(C5) $\quad \alpha_i \cup \alpha_j \simeq \lambda b.b \in \alpha_i \vee b \in \alpha_j,$

(C6) $\quad \alpha_i \cap \alpha_j \simeq \lambda b.b \in \alpha_i \;\&\; b \in \alpha_j,$

(C7) $\quad \neg\alpha \simeq \lambda b \sim b \in \alpha.$

The bundle calculus is developed from the axioms and rules of PC and subject analogues of those of the predicate calculus plus the following:

(S1) $\quad .f \ni \theta gP \equiv f/gP,$

(S2) $\quad .f \ni a \equiv fa,$

(S3) $\quad .x_i \sqsubset x_j \equiv \forall g.g \ni x_i \supset g \ni x_j,$

(S4) $\quad .x_i \approx x_j \equiv .x_i \sqsubset x_j \;\&\; x_j \sqsubset x_i,$

(S5) $\quad x_i \sqcup x_j \approx \theta g.g \ni x_i \vee g \ni x_j,$

(S6) $\quad x_i \sqcap x_j \approx \theta g.g \ni x_i \;\&\; g \ni x_j,$

(S7) $\quad \lrcorner x \approx \theta g \sim g \ni x.$

A logic embodying both calculi requires additional axioms incorporating scope requirements: (M1) $.\alpha \ni x \equiv x/b\, b \in \alpha$, (M2) $.x \in \alpha \equiv \alpha/g\, g \ni x$, which have useful corollaries: $.\alpha \ni a \equiv a \in \alpha$ and $.x \in f \equiv f \ni x$. A natural language interpretation, in which scope is indicated by commas, has application in the analysis of zeugmas.

The logic can be extended to include an individual calculus of things such as that of Leonard and Goodman and its mirror image a taxonomic calculus of attributes. The calculus of Goodman's Structure of Appearance can be subsumed into the bundle calculus.

The development of second order predicate (subject) calculi requires the introduction of class (bundle) variables with quantification ranging over classes (bundles) in general.

MOSTOWSKI, ANDRZEJ. *A transfinite sequence of ω-models.*

Denote by A_2 the system of 2nd order arithmetic as described in Mostowski-Suzuki, ***Fundamenta Mathematicae***, vol. 65 (1969), pp. 83–93. ω-models of this sytem will be identified with the families of their sets. We denote by M_{pr} the "principal" model containing all sets of integers and by F the family of all denumerable ω-models which are elementarily equivalent to M_{pr}. A set C of integers is called a code of a denumerable family M of sets of integers if M coincides with the family of sets $C_n = \{m: 2^n(2m-1) \in C\}$, $n = 1, 2, \cdots$. We say that MeN if N contains a code of M.

Using methods similar to those of the quoted paper one shows the following

THEOREM. *There exists a family $F_1 \subseteq F$ with the properties:* (i) *If $M, N \in F$, then either $M \prec N$ and $M e N$ or $N \prec M$ and $N e M$*; (ii) *the order type of the relation $\prec$ in F is η where η is the type of rational numbers.*

COROLLARY 1. *There is a set of sets of integers which is ordered in type η by the relation "to be hyperarithmetical in".*

COROLLARY 2. *There is a family $F_2 \subseteq F$ which is ordered in type $\eta \cdot \omega_1$ by the relation e.*

We say that an ω-model M has property (P) if for every set X in M there is an ω-model N such that $X \in N \prec M$ and $N e M$.

COROLLARY 3. *For every set X of integers there exists an ω-model M in F such that $X \in M$ and M has property (P).*

SCHUMM, GEORGE F. *Trees, bouquets, and extensions of* S4.

We consider extensions of S4 by the axioms:

A. $\Box(\Box(p \supset \Box p) \supset p) \supset (\Box\Diamond\Box p \supset p)$,

B. $\Box(\Box(p \supset \Box p) \supset p) \supset p$,

C_n. $p_1 \supset \Box(\sim p_1 \supset (p_2 \supset \cdots (p_n \supset \Box(\sim p_n \supset (\Diamond q \supset \Box\Diamond q)\cdots))))$,

D_n. $p_1 \supset \Box(\sim p_1 \supset (p_2 \supset \cdots (p_n \supset \Box(\sim p_n \supset (q \supset \Box q)\cdots))))$,

E_m. $\Box\Diamond \bigvee_{1 \le i < j \le 2^{m+1}} \Box(p_i \equiv p_j)$,

proving each such system decidable and to be complete relative to an appropriate relational modelling. Of these systems, S4B, S4BD_1, and S4AC_1 are equivalent to Sobociński's Kl.1, Kl.2, and Zeman's S4.04, respectively.

A relational model $\mathfrak{A} = \langle W, R \rangle$ is called a *bouquet* if $W = X \cup \bigcup_{x \in Q} \mathscr{C}_x$ with $\mathscr{C}_x \cap X = \{x\}$ and R the smallest reflexive and transitive relation on W such that $\mathfrak{B} = \langle X, R \restriction X \rangle$ is a finite tree and R is universal on $\mathscr{C}_x$ for each x in Q, the set of endpoints of $\mathfrak{B}$. $\mathscr{C}_x$ is a *blossom* of $\mathfrak{A}$ and the elements of $\mathscr{C}_x$ are its *petals*. We say that $\mathfrak{A}$ is an *n-bouquet* if every branch of $\mathfrak{B}$ is of order type $\le n + 1$.

THEOREM. S4A (S4AC_n, S4AE_m, S4AC_nE_m, S4B, S4BD_n) *is determined by the class of finite bouquets* (*n-bouquets, bouquets whose every blossom contains at most m petals, n-bouquets whose every blossom contains at most m petals, trees, trees whose every branch is of order type* $\le n + 1$).

COROLLARY. $\text{S4A} = \bigcap_{1 \le i < \omega} \text{S4AC}_i$, $\text{S4AC}_n = \bigcap_{1 \le i < \omega} \text{S4AC}_n\text{E}_i$, $\text{S4AE}_m = \bigcap_{1 \le i < \omega} \text{S4AC}_i\,\text{E}_m$, *and* $\text{S4B} = \bigcap_{1 \le i < \omega} \text{S4BD}_i$.

SCHUMM, GEORGE F. *Finite limitations on some extensions of* T.

The Feys-von Wright system T is known to be determined by the class of finite reflexive relational models, while the class of finite reflexive and symmetric models determines its *Brouwersche* extension B. Letting T_n and B_n be the results of enriching T and B, respectively, with the Dugundji axiom

$$\bigvee_{1 \le i < j \le 2^n + 1} \Box(p_i \equiv p_j)$$

we show that $T_n(B_n)$ is a proper extension of $T_{n+1}(B_{n+1})$ and

THEOREM. $T_n(B_n)$ *is determined by the class of finite reflexive* (*reflexive and symmetric*) *relational models* $\langle W, R \rangle$ *such that for each* $w \in W$ *there are at most n elements x of W for which* wRx.

COROLLARY. $T = \bigcap_{1 \le i < \omega} T_i$ *and* $B = \bigcap_{1 \le i < \omega} B_i$.

Suppose S is any one of the following extensions of T: S4, S4.2, S4.3, S5, Sobociński's S4.1, S4.4, K1, K2, K3, K1.1, K2.1, K3.1, K1.2, K3.2, Prior's D, and Zeman's S4.3.2, S4.04. Then if S is extended with the Dugundji axiom, S_n is a proper extension of S_{n+1} and

THEOREM. *For each* S *there is a class C of finite relational models such that C determines* S *and* S_n *is determined by the class of n-element models in C.*

COROLLARY. $S = \bigcap_{1 \le i < \omega} S_i$.

This generalizes an analogous result originally obtained for S5 by Scroggs (this JOURNAL, vol. 16, pp. 112–120) using the 2^n-valued Henle matrices, and enables us to axiomatize several many-valued matrices which have appeared in the literature. The $K3.1_n$'s axiomatize the 2^n-valued matrices mentioned by Prior on pp. 15–16 of ***Time and modality***, Oxford, 1957, while $K1.2_3$ and $K3.2_3$ axiomatize eight-valued matrices constructed by Prior (***Notre Dame journal of formal logic***, vol. 5, p. 299) and Zeman (*ibid.*, vol. 9, p. 297). $S4_2$ axiomatizes a sixteen-valued matrix due to Sobociński (*ibid.*, vol. 11, p. 350, matrix 10) and is deductively equivalent to his system V1.

SEGERBERG, KRISTER. *On the extensions of S4.4.*

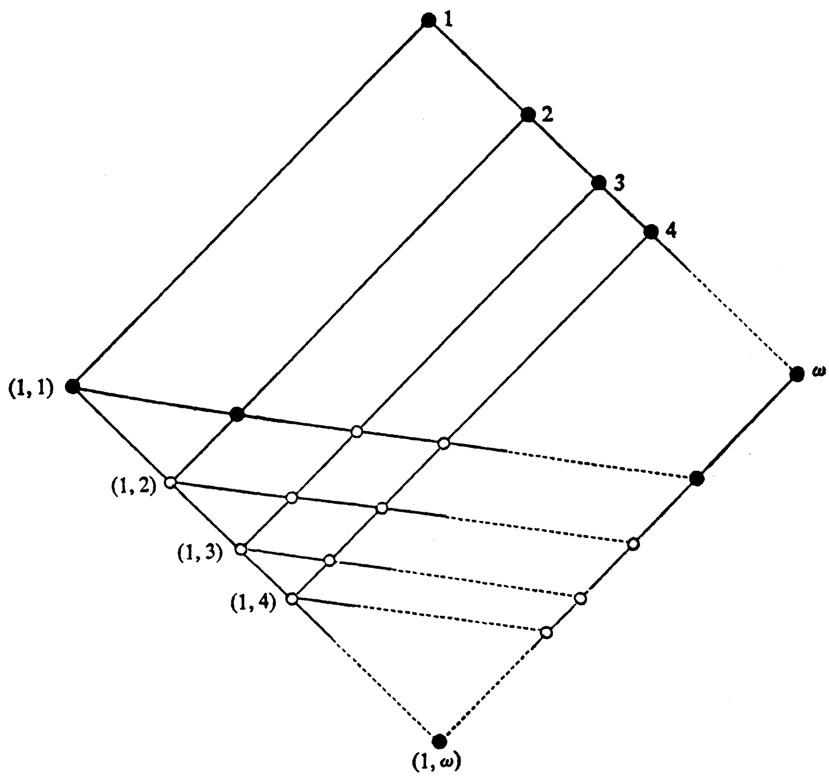
We use the terminology of [2]. For definitions of the modal logics mentioned below, see [1] and [4]. We assume the identifications $n = \{0, 1, \cdots, n - 1\}$ and $\omega = \{0, 1, \cdots\}$. An *index* (*of length* 2) is an ordered couple (t_1, t_2) such that $t_1, t_2 \le \omega$. Every index *induces* a frame $\langle U, R \rangle$, namely that for which $U = \{(m, n): m = 0 \;\&\; n < t_1, \text{ or } m = 1 \;\&\; n < t_2\}$ and $(m, n)R(m', n')$ iff $m \le m'$. A logic is said to have index (t_1, t_2) if it is determined by the frame induced by (t_1, t_2). A logic is an *index logic* if it has an index. It is clear that *S5* has index ω.

Scroggs's Theorem says, essentially, that the only proper extensions of *S5* are the index logics n, with $n < \omega$.

Using the index terminology the extensions of *S4.4* can be completely described in a simple manner; the chart gives the structure of the entire set of these extensions. As usual, a logic is strictly weaker than another if it is connected to the other logic by a rising line. The intersection of two logics is the strongest common sublogic. For every index logic its index is indicated. A logic which is not an index logic is the intersection of two index logics, and is determined by the two-element set of indices of these logics. Every extension of *S4.4* is normal. The logics already described in the literature are represented by filled spheres. Apart from *S5* and its extensions, they and their corresponding index or set of indices are:

S4.4	$(1, \omega)$
K4	$(1, 1)$
V1	$\{(1, 1), 2\}$
S4.7	$\{(1, 1), \omega\}$.

(Note that by *K4* is meant the system so designated by Sobociński. In terms of [3], that system is the same as *S4.3GA*$_2$.) There are no intercalary logics except where the lines are broken. In particular, *S4.7* is the strongest extension of *S4.4* to be properly included in *S5*, and *S4.7* is the strongest common sublogic of *K4* and *S5*.



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WEBB, PHILIP. *A pair of primitive rules for the sentential calculus.*

The system:

$$
\begin{array}{llcll}
/I & p & \qquad & /E & p \mid q/r \\
 & q & & & p \\
 & r & & \vdash & q \\
 & \vdash r/q \mid p & & & r
\end{array}
$$

is easily shown complete.

The sytem can be proved unique (with minor variants) using a tautology A, containing C^{**}, C'^{**} s.t. if A' is constructed from A by replacing C^{**}, C'^{**} by B, B' where $-(B \prec B')A'$ is not tautologous; C^{**}, C'^{**} are constructed from $C^{\iota*}$, $C'^{\iota*}$ as C^{ι}, C'^{ι} are from C_j, C'_j, and $C^{\iota*}$, $C'^{\iota*}$ lie within $>x/s$ in C^{**}, C'^{**}; $C^{\iota*} \in \Sigma'(C^{\iota})$ if i is odd or $\Sigma(C^{\iota})$ if i is even, and $C'^{\iota*}$ is similar but reversing 'odd' and 'even', and C^{ι}, C'^{ι} lie within $>x/s$ in $C^{\iota*}$, $C'^{\iota*}$; $C^{\iota} = C_1^{v\iota}/\cdots/C_n^{v\iota}$ ($C_j^{v\iota}$ is a variable; $n - 1 > x$; group to right) or is got from it by replacing 1 or more $C_j^{v\iota}$ by $C_j^{\iota} \in \Sigma(C_j^{v\iota})$ if j is even or $\Sigma'(C_j^{v\iota})$ if j is odd, and C'^{ι} is similar but reversing 'odd' and 'even'; $\Sigma(E)$ is the sequence consisting of E and all WFFs got by writing $E/G \mid E/H$ for E in an earlier member, and $\Sigma'(E)$ is similar but writing $E/E \mid G$ (so if $J \in \Sigma(E) J \prec E$, and if $J \in \Sigma'(E) E \prec J$); and A satisfies other minor conditions. It can be shown that for almost any other pair of natural-deductive rules where no variable lies within $>x/s$, A is not derivable. For the rules must allow the reduction by 2 at a time of the number of $/s$ within which $C_j^{v\iota}$ lies till it lies within $0/s$. So there must be a rule of detachment, with one premiss and one line of its conclusion a single variable; whence it is easy to show the rules must resemble almost exactly those above.

WHERRITT, DR. ROBERT C. *First-order equality logic with weak existence assumptions.*

We formulate and prove completeness theorems for several classes of first-order logics with equality and function symbols (including individual constants as 0-ary function symbols) whose existence assumptions diminish in strength from the standard ones ($\exists x(x = t)$ is provable for any term t) to the weakest ones (no existential formulas are provable). The semantics is based on a generalization of Tarski's notion of an interpretation called a *semirealization* in which there is a nonvoid universe S, a *domain* $D \subset S$, and a *semantimorphism* σ which associates semantic objects with syntactical objects so that: (i) for each n-ary predicate letter P, $\sigma P \subset S^n$, (ii) for each n-ary function letter f, σf is a *partial* function from S^n to S, (iii) there is a nonvoid set R with $D \subset R \subset S$ such that free variables range over R while bound variables are restricted to range within D, (iv) σ restricted to formulas is a two-valued homomorphism with respect to the logical functors. A semirealization Q is *strong* if $D = R$, and Q is called a *full realization* if $D = S$ and each σf is a total function.

THEOREM 1. *The standard propositional rules and axioms, the rule* $\forall$Intro, *the quantifier rules* $\vdash \forall y(\forall x A \Rightarrow A(y/x))$ *and* $\forall x(A \Rightarrow B) \vdash \forall x A \Rightarrow \forall x B$, *and the standard equality rules are all valid in every semirealization. Conversely, every formula true in all semirealizations is provable by the rules given above.*

THEOREM 2. *Besides the rules and axioms above,* $\forall x A \vdash A(y/x)$ *is valid in the class of all strong realizations. Conversely, every formula true in this class is provable from the rules and axioms given above.*

THEOREM 3. *Besides the rules and axioms above, the rule* $\forall$Elim $\forall x A \vdash A(t/x)$ *for every term* t *is valid in the class of all full realizations. Conversely, any formula true in this class is provable from the rules and axioms given above.*