A. For a fixed number n, what is the number of maximal subsets of F_n?

B. Can a subset precomplete in F_n be infinitely generated?

C. Given an infinitely generated set $F \subset F_n$, let a maximal set F' be constructed. Is the extension F' always unique?

It is shown that maximal subsets of $L(n)$ can be constructed such that for $L(n)$ the answer to question C is negative while the answer to question B is positive. An interesting conjecture is given also regarding an answer to question A. The paper contains several theorems which shed additional light on the little known properties of infinitely generated sets and the author frames some suggestive conjectures which will serve to stimulate further research on such sets. ATWELL R. TURQUETTE

LENNART ÅQVIST. *Interpretations of deontic logic.* ***Mind,*** n.s. vol. 73 (1964), pp. 246–253.

Let DL be, essentially, the deontic logic of von Wright XVIII 174, except that propositional variables are used in place of act-variables and Op has the sense "the state of affairs described by p is obligatory." The author proposes five alternative interpretations of DL: Op may be read as (i) "I command that p"; (ii) "I wish that p"; (iii) "I promise that p"; (iv) "I decide that p"; (v) "I intend that p." We may further distinguish internal and external interpretations of Op: "I command that p," "I wish that p," etc., can be read *internally* as expressing a command, a wish, etc., or they can be read *externally* as expressing the proposition that the speaker does command, wish, etc., that p. Thus "I command that p and I command that not-p" is contradictory read internally, but consistent read externally. For the axiom $Op \supset \sim O \sim p$ to hold for these interpretations, therefore, Op has to be taken internally. It follows that $\sim Op$ requires internal interpretation also: e.g. "I don't promise that p" expresses something that stands in the same relation to a promise that p that $\sim p$'s being permitted stands in to p's being obligatory. Finally, the author defends his view that the logic of the five given interpretations is analogous to that of Op and not to that of Pp ($= \sim O \sim p$). *Comments.* This paper is suggestive, but much more remains to do: it needs bringing out that some of these interpretations ((i)–(iii)?) are "performative" and others not; also, if Op does not express a proposition, then the symbol $\sim$ in $\sim O \sim p$ is ambiguous — indeed two negation signs are needed, and similarly for the other truth-functional operators (compare the Hofstadter-McKinsey imperative logic in V 41).

E. J. LEMMON

SAUL A. KRIPKE. *Semantical analysis of modal logic I. Normal modal propositional calculi.* ***Zeitschrift für mathematische Logik und Grundlagen der Mathematik,*** vol. 9 (1963), pp. 67–96.

In his 1959 abstract *Semantical analysis of modal logic* (this JOURNAL, vol. 24 (1959), pp. 323–324), the author announced completeness, decidability, and other results for a wide variety of modal calculi. In the present paper he describes his semantical theory, supplies proofs of the announced results for certain propositional modal calculi, and continues the exposition, begun in *A completeness theorem in modal logic* (this JOURNAL, vol. 24 (1959), pp. 1–14), of his adaptation of Beth's semantical tableaux to modal logic. The calculi treated are Gödel-Feys-von Wright's M, the *Brouwersche* system B, and Lewis's S4 and S5. All contain the axiom schemes A0, A1 (the author inadvertently omits A0), and the rules R1 and R2. In addition, M contains A2, B contains A2 and A3, S4 contains A2 and A4, and S5 contains A2 and A5 (or equivalently, A2, A3, and A4). (A0) A, if A is a tautology; (A1) $\Box(A \supset B) \supset (\Box A \supset \Box B)$; (A2) $\Box A \supset A$; (A3) $A \supset \Box \sim \Box \sim A$; (A4) $\Box A \supset \Box \Box A$; (A5) $\sim \Box A \supset \Box \sim \Box A$; (R1) if $\vdash A \supset B$ and $\vdash A$, then $\vdash B$; (R2) if $\vdash A$, then $\vdash \Box A$.

The basic semantical innovation lies in the notion of one world being *possible relative to* another. $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ is a model structure (m.s.) iff K is a non-empty set, $\mathsf{G} \in \mathsf{K}$, and

$\boldsymbol{R}$ is a relation defined on K. Here K is thought of as the set of (possible) worlds of the m.s., G is thought of as the actual world, and "$\mathsf{H}\ \boldsymbol{R}\ \mathsf{H}'$" is read "$\mathsf{H}'$ is possible relative to H." If $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ is a m.s., $((\mathsf{G}, \mathsf{K}, \boldsymbol{R})\Phi)$ is a model iff to each formula A and each $\mathsf{H} \in \mathsf{K}$, Φ assigns a truth-value in accordance with the usual requirements for sentential connectives (e.g. $\Phi({\sim}A, \mathsf{H}) = \mathsf{T}$ iff $\Phi(A, \mathsf{H}) = \mathsf{F}$) and the special requirement: $\Phi(\Box A, \mathsf{H}) = \mathsf{T}$ iff $\Phi(A, \mathsf{H}') = \mathsf{T}$ for all $\mathsf{H}' \in \mathsf{K}$ such that $\mathsf{H}\ \boldsymbol{R}\ \mathsf{H}'$. A is true in $((\mathsf{G}, \mathsf{K}, \boldsymbol{R})\Phi)$ iff $\Phi(A, \mathsf{G}) = \mathsf{T}$.

The completeness theorems were produced by the insight that, "the reduction axioms of classical modal logic reduce to simple properties...of the relation $\boldsymbol{R}$." Thus, call a model $((\mathsf{G}, \mathfrak{K},'\boldsymbol{R})\Phi)$ an M-model (also, a normal model) if $\boldsymbol{R}$ is reflexive, a B-model if $\boldsymbol{R}$ is reflexive and symmetrical, an S4-model if $\boldsymbol{R}$ is reflexive and transitive, and an S5-model if $\boldsymbol{R}$ is an equivalence relation. Note that $((\mathsf{G}, \mathsf{K}, \boldsymbol{R})\Phi)$ is an S5-model if and only if it is both a B-model and an S4-model. The main theorem states that A is provable in M, B, S4, or S5 if and only if it is true in all of the corresponding models. (Finding a non-trivial characterization of the properties of a relation $\boldsymbol{R}$ which can be so expressed by modal axioms seems to the reviewer an interesting open problem.) The method of proof yields the slightly stronger, but not surprising, result that we could limit consideration to tree-M, B, S4, or S5 models. A m.s. $(\mathsf{G}, \mathsf{K}, \boldsymbol{S})$ is a tree iff the converse of $\boldsymbol{S}$ is a function and G is the unique element of K such that for all $\mathsf{H} \in \mathsf{K}$ *other than* G (omitted by the author) $\mathsf{G}\ \boldsymbol{S}^*\ \mathsf{H}$, where $\boldsymbol{S}^*$ is the ancestral of $\boldsymbol{S}$. A model $((\mathsf{G}, \mathsf{K}, \boldsymbol{R})\Phi)$ is a tree-M, B, S4, or S5 model iff for some $\boldsymbol{S}$ $(\mathsf{G}, \mathsf{K}, \boldsymbol{S})$ is a tree and $\boldsymbol{R}$ is the smallest relation which includes $\boldsymbol{S}$, has the appropriate properties, and has K as its field. (The author neglects the tree $(\mathsf{G}, \{\mathsf{G}\}, \Lambda)$ and so incorrectly omits the last clause.) In the last third of the paper the tableau constructions are made to yield decision procedures (which the author claims to be the simplest in the literature), and the completeness theorem is shown to yield denumerable characteristic matrices for each of the systems considered. In the final paragraph, the author mentions that for the system axiomatized by A0 and A1 with rules R1 and R2, call it K, we simply drop the requirement that $\boldsymbol{R}$ be reflexive, and he remarks that systems of this type are required for deontic logic. The system K has a separate interest as the weakest system which can be treated without modification of the author's methods. Results for K comparable to those obtained for M, B, S4, and S5 are already implicit in the proofs given. A number of other interesting points are touched upon in the course of the paper.

Although the author extracts a great deal of information from his tableau constructions, a completely rigorous development along these lines would be extremely tedious. As a consequence a number of small gaps must be filled by the reader's geometrical intuition, for example in verifying that the construction can be developed so that every line in every tableau will have the appropriate rule applied to it at some point. The dangers inherent in relying on intuition are illustrated by the author's need to correct a fallacious proof in *A completeness theorem in modal logic* where nodes and branches of the construction are conflated, and his erroneous claim that his "S-formulations" of the rules are equivalent to the "R-formulations." In the latter connection there seems to be some special difficulty about the S-rule Y1 for S5; the author criticizes another writer's faulty version of the rule; but his own formulation also requires amendment, by adding the clause: put $\Box A$ on the left of any tableau $\mathfrak{t}_3$ such that $\mathfrak{t}_3\ S\ \mathfrak{t}_1$. The proofs of the decision procedures seemed to the reviewer excessively intuitive even within the allowable space, and the proof for S4 contains an error (which can be corrected if "are *equal* to" is replaced by "are *contained* in" in the definition of "*saturated*" page 89, line 32).

The reviewer believes that future research will bring considerably simpler more rigorous proofs which avoid the tableau technique. In fact the interesting half of the main theorem can be established by using the technique of Henkin XV 68. Let A

be a non-theorem and consider the model $((\mathsf{G}, \mathsf{K}, \boldsymbol{R}), \Phi)$ where K is the set of all complete, deductively consistent sets of formulas; $\mathsf{G} \in \mathsf{K}$ and $\sim A \in \mathsf{G}$; $\mathsf{H}_1 \boldsymbol{R} \mathsf{H}_2$ iff for every formula D such that $\Box D \in \mathsf{H}_1$, $D \in \mathsf{H}_2$; and for each sentence letter $\boldsymbol{P}$, $\Phi(\boldsymbol{P}, \mathsf{H}) = \mathsf{T}$ iff $P \in \mathsf{H}$. It is easily established that the axioms of the system in question insure that $\boldsymbol{R}$ has the appropriate properties so $((\mathsf{G}, \mathsf{K}, \boldsymbol{R}), \Phi)$ is a model of the appropriate kind. In the inductive proof that for an arbitrary formula B and $\mathsf{H} \in \mathsf{K}$, $\Phi(B, \mathsf{H}) = \mathsf{T}$ iff $B \in \mathsf{H}$, the only non-trivial case is where B has the form $\Box C$, and $\Phi(\Box C, \mathsf{H}) = \mathsf{T}$. In this case C is in every complete deductively consistent set which contains all D such that $\Box D \in \mathsf{H}$. Hence there are such $D_1, \ldots, D_n$ $(n \geqq 1)$ for which $D_1 \wedge \ldots \wedge D_n \supset C$ is a theorem. But then $\Box D_1 \wedge \ldots \wedge \Box D_n \supset \Box C$ is also a theorem and thus $\Box C \in \mathsf{H}$. Hence since $\sim A \in \mathsf{G}$, $((\mathsf{G}, \mathsf{K}, \boldsymbol{R}), \Phi)$ is the desired countermodel. The idea of adapting Henkin's technique to modal systems was first suggested to the reviewer by Dana Scott, although in the context of a somewhat different semantical theory. It should also be noted that this argument is foreshadowed in Kanger's XXIII 37, theorem 8 (corrected in XXIII 38).

Algebraic and matrix interpretations of modal logic now abound, but very little progress has been made in developing a "natural" semantical theory for modal logic since the appearance of Carnap's XIII 219 (1946) and XIV 237 (1947). The author's semantical conceptions constitute a significant contribution to this field. Credit for these conceptions must be shared with Kanger (see his 1957 booklet XXIII 37) and Hintikka (whose results were first published in XXXI 122(1)), both of whom earlier introduced a relation between worlds and explicitly stated completeness theorems for M, S4, and S5 in terms of the reflexivity, reflexivity and transitivity, and reflexivity transitivity and symmetry of this relation. Kanger's booklet (which, in the opinion of the reviewer, has not received the attention it deserves) contains the earliest published discussion but is marred by a defect in the formulation of the completeness theorems. His formulation implies that all theorems hold *only* in models $((\mathsf{G}, \mathsf{K}, \boldsymbol{R}), \Phi)$ where $\boldsymbol{R}$ has the appropriate properties. But the best that can be said in this case is that the function Φ is unaffected if $\boldsymbol{R}$ is replaced by its closure under the appropriate properties. The semantical theories of Kanger and Hintikka take rather different forms from that of the author, and the reviewer slightly prefers the author's. It should also be mentioned that in a paper read at U.C.L.A. in 1955 and later published as *Logical necessity, physical necessity, ethics, and quantifiers* (***Inquiry***, vol. 3 (1960), pp. 259–269), Montague suggested the interpretation of modal calculi in terms of a relation between worlds, although he did not anticipate the particular semantics for the systems here under consideration.

Questions of historical priority and the possibility of improving the arguments do not diminish the reviewer's judgment that the present paper is among the most important contributions to the study of modal logic.

The paper contains a fair number of misprints. The following corrections should be made: page 71, line 13, substitute '$\boldsymbol{R}$' for '$\boldsymbol{R}'$', line 14, '$\boldsymbol{R}'$' for the first '$\boldsymbol{R}$', and line 16, '$\boldsymbol{R}'$' for the last '$\boldsymbol{R}$'; page 73, line 15, 't'' for the last 't'; page 79, line 37, '$\Box B$' for 'B', and line 38, 'for some descendant, t'', of t, $t'' \boldsymbol{R} t'$' for '$t \boldsymbol{R} t'$'; page 87, lines 31 and 32, 't_i'' for 't_i', and lines 36 and 42, 't'' for 't'; page 88, line 11, 't_i'' for 't_i', and line 37, 't_1'' for 't_i''; page 89, line 11, 'model' for 'example' and 'contain' for 'be'; page 91, in the diagram delete the last three lines of t_3; page 94, line 8, 'denumerably' for 'countably,' and line 9, '$\boldsymbol{S}'$' for '$\boldsymbol{S}$'; page 95, line 1, 'G'' for the last 'G'.

DAVID KAPLAN

JAAKKO HINTIKKA. *Modality and quantification.* ***Theoria*** (Lund), vol. 27 (1961), pp. 119–128.

JAAKKO HINTIKKA. *The modes of modality.* ***Proceedings of a Colloquium on***