Nominalistic Set Theory

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Introduction

By means that meet the standards of nominalism set by Nelson Goodman (in [1], section II, 3; and in [2]) we can define relations that behave in many ways like the membership relation of set theory. Though the agreement is imperfect, these pseudo-membership relations seem much closer to membership than to its usual nominalistic counterpart, the part-whole relation. Someone impressed by the diversity of set theories might regard the theories of these relations as peculiar set theories; someone more impressed by the non-diversity of the more successful set theories—ZF and its relatives—might prefer not to. This verbal dispute does not matter; what matters is that the gap between nominalistic and set-theoretic methods of construction is narrower than it seems.

Preliminaries

A finitistic nominalist's world might consist of an enormous hypercubical array of space-time points, together with all wholes composed of one or more of those points. Each point in the array is next to certain others; nextness is a symmetric, irreflexive, intransitive relation among the points. We can (but the nominalist cannot) describe the array of points and the nextness relation more precisely by stipulating that the points can be placed in one-to-one correspondence with all the quadruples $<x, y, z, t>$ of non-negative integers less than or equal to some very large integers x_{max}, y_{max}, z_{max}, t_{max}, respectively in such a way that one point is next to another iff the corresponding quadruples are alike in three coordinates and differ by exactly one in the remaining coordinate. We take this array for the sake of definiteness; but what follows does not depend on the exact shape and structure of the array. Large orderly arrays of various other sorts would do as well.

We will employ two primitive predicates:

x is part of y
x is next to y.

Both appear in [1] (sections II, 4 and X, 12), though not as primitive; so we can take them to be safely nominalistic. We will be concerned to define other predicates by means of these two primitives and first-order predicate logic with identity. (We will use English, but in such a way that it will be clear how to translate our definitions into logical notation.)

Using the part-whole predicate, we can define the various predicates of the calculus of individuals. We shall use them freely henceforth. In particular, let us define an *atom* as something having no parts except itself, so that atoms are the points in our array; and let us call x an *atom of* y iff x is an atom and x is part of y. Let us also say that x is the *universe* iff every atom is part of x.

From time to time we shall use set theory, but only in an auxiliary role. Set-theoretical definitions will not be part of our nominalistic construction. When we speak simply of sets or membership, we are to understand that standard sets or membership (say, in the sense of ZF) are meant. When we wish rather to speak of our nominalistic sets or membership, we will use subscripts or the prefix "pseudo-".

Let us say that x *touches* y iff some atom of x is next to some atom of y.

Let us call x the *interior* of y iff the atoms of x are all and only those atoms of y that are not next to any atoms except atoms of y. Not everything has an interior; a single atom, or a one-dimensional string of atoms, or a two-dimensional sheet of atoms, or a three-dimensional solid of atoms does not. Only the universe is identical with its interior.

Let us call x the *closure* of y iff the atoms of x are all and only those atoms that either are or are next to atoms of y. Everything has a closure. Only the universe is identical with its closure.

We might guess that everything that has an interior is the closure of its interior and that everything is the interior of its closure. Both guesses are wrong. Let us call something *stable* iff it has an interior and is the closure of its interior, and *solid* iff it is the interior of its closure. Not everything that has an interior is stable: if x is stable, y has no interior, and x does not touch y, then x + y has an interior but is not stable. Not everything is solid: if x is solid and has an interior and y is an atom of the interior of x, then x-y is not solid; the interior of the closure of x-y is x, since y is restored in taking the closure and remains when we take the interior. Note that the interior of any stable thing is solid and the closure of any solid thing is stable.

Let us call x and y *almost identical* iff each of the differences x-y and y-x either does not exist or has no interior. The relation of almost-identity is reflexive and symmetric but not transitive.

Let us call x *connected* iff, whenever y is a proper part of x, y touches x-y. The universe is connected. Let us call x *well-connected* iff the interior of x is connected. Let us call x a *maximal connected part of* y iff x is a connected part of y but x is not a part of any other connected part of y.

If x is the interior of a connected thing y, let us call y a *connection* of x and let us call x *connectible*. All solid connected things are connectible, since the closure of such a thing is one connection of it. With a few exceptions, solid disconnected things are connectible. We can make a connection of x by taking its closure and adding strings of atoms running from one maximal connected part of the closure of x to another. This can be done provided there are places to put the connecting strings such that every atom of the strings and every non-interior atom of the closure of x remains a non-interior atom of the closure-plus-strings. Then when we take the interior of the closure-plus-strings, the strings disappear and we get the interior of the closure—that is, since x is solid, we get x.

Something x could be solid but not connectible if, for instance, x was flattened out along some of the edges of our array. It could happen then that every non-interior atom of the closure of x was next to only one atom other than the atoms of the closure. Then we could not attach strings without converting an atom at the point of attachment from a non-interior atom of the closure to an interior atom of the closure-plus-strings. Since any such atom will be left behind when we take the interior, we will not recover x; so the closure-plus-strings is not a connection of x. Such cases, however, seem to occur only under rather special conditions.

Membership$_1$

Let us call x a *member$_1$* of y iff x is a maximal connected part of y. The theory of membership$_1$ resembles the usual theory of membership: that is, set theory. We have a principle of extensionality:

E1. If all and only members_1 of x are members_1 of y, then x and y are identical.

We have a restricted principle of comprehension:

C1. There is something z having as its members_1 all and only those things x such that __x__, provided that: (1) whenever __x__, x is connected; (2) no two objects x such that __x__ touch; (3) there is something x such that __x__.

(Here and henceforth, "__x__" is schematic for any formula of our language in which "x" apears free.) We also have a restricted principle of *Aussonderung:*

A1. There is something z having as its members_1 all and only those members_1 x of y such that __x__, provided that there is at least one such member_1 of y.

Indeed, for any non-empty set S of connected, non-touching things, there is a unique thing z whose members_1 are all and only the members of S: namely, the sum of the members of S.

Like membership, and unlike the part-whole relation, membership_1 is irreflexive. Disconnected things are not members_1 of themselves or of anything else. Only connected things are members_1 of anything, and they are their own sole members_1 (*cf.* Quine's identification of individuals with their unit sets in [3]). Like the part-whole relation, and unlike membership, membership_1 is transitive. But in a trivial way: if x is a member_1 of y and y is a member_1 of z, then x is a member_1 of z because x and y are identical.

Like the part-whole relation, and unlike membership, membership_1 satisfies Goodman's criterion for a nominalistic generating relation: no distinction of entities without distinction of content ([2], section 2). It is not clear to me whether membership_1 is what Goodman calls a "generating relation"; but if it is one, then it is a nominalistic one. Let the $\textit{content}_1$ of x be the set of things which bear the ancestral of membership_1 to x but have no members_1 different from themselves. (This set-theoretical definition of a set is, of course, no part of our nominalistic construction.) No two different things have the same content_1.

Membership$_2$

The members$_1$ of something cannot touch, since if two things touch, they cannot both be maximal connected parts of anything. However, let us call x a *member*$_2$ of y iff x is the closure of a maximal connected part of y and every maximal connected part of y is solid. Then for any non-empty set S of stable, well-connected, non-overlapping things, even if these things touch, there is a unique thing whose members$_2$ are all and only the members of S: namely, the sum of the interiors of members of S.

We have a principle of restricted extensionality, applying only to those things that have members$_2$:

E2. If all and only members$_2$ of x are members$_2$ of y and if x and y have members$_2$, then x and y are identical.

We have principles of restricted comprehension and restricted *Aussonderung:*

C2. There is something z having as its members$_2$ all and only those things x such that __x__, provided that: (1) whenever __x__, x is stable and well-connected; (2) no two objects x such that __x__ overlap; (3) there is something x such that __x__.

A2. There is something z having as its members$_2$ all and only those members$_2$ x of y such that __x__, provided that there is at least one such member$_2$ of y.

Any stable connected thing is a member$_2$ of something, but nothing else is. Anything whose maximal connected parts are solid has at least one member$_2$, but nothing else does. The proviso that a pseudo-set must consist of solid maximal connected parts is required to ensure extensionality[1]. Let x consist of solid maximal connected parts; let y be an atom of the interior of x; then exactly the same things are closures of maximal connected parts of x and of x-y. But x-y has a non-solid maximal connected part and hence has no members$_2$. Note that a pseudo-set consisting of solid maximal connected parts may itself not be solid.

Membership$_2$ is almost, but not quite, irreflexive. Only one thing is a member$_2$ of itself: the universe. Membership$_2$ is almost intransitive, with the same exception: if x is a member$_2$ of y and

[1] I owe this observation to Oswaldo Chateaubriand.

y is a $member_2$ of z, then x is not a $member_2$ of z unless x and y both are the universe.

Though nominalistically defined, $membership_2$ is not a nominalistic generating relation according to Goodman's criterion. Let the *content*$_2$ of x be the set of things which bear the ancestral of $membership_2$ to x but have no $members_2$ different from themselves. Then there are distinct things having the same $content_2$. For instance, if x and y are any two solid connected things such that the closures, the closures of the closures, etc. of x and y are solid, then x and y have exactly the same $content_2$: the unit set of the universe.

Not only the letter but the spirit of Goodman's criterion is violated. We get different entities corresponding to different ways of dividing up the same stuff, provided we consider only divisions of it into stable, well-connected, non-overlapping parts. Let the result of enclosing a list in braces denote the thing having as $members_2$ all and only the listed things. Suppose we have three things x, y, and z which touch each other but do not overlap. Assume x, y, z, x + y, y + z, x + z, x + y + z all are stable and well-connected; and assume x, y, and z do not together exhaust the universe. Then we have the following six distinct things:

$$x + y + z \qquad \{x + y + z\} \qquad \{x, y, z\}$$
$$\{x + y, z\} \qquad \{x, y + z\} \qquad \{x + z, y\}$$

Yet we have not matched the standard set theorist's power to generate entities. We do not have, for instance, these:

$$\{x + y, y + z\} \qquad \{x, x + y, x + y + z\}$$

If land were made out of our atoms, we could now handle one of Goodman's problematic sentences in [1] (section II, 3): "At least one group of lots into which it is proposed to divide this land violates city regulations."

We can thus have touching pseudo-members, but not overlapping ones. We might proceed further: call x a *member*$_2^0$ of y iff x is a $member_2$ of y; call x a *member*$_2^{n+1}$ of y iff x is a $member_2$ of a $member_2^n$ of y. (This is a definition-schema for a sequence of 2-place predicates, not a recursive specification for one 3-place predicate.) In this way we could get things with overlapping pseudo-members, provided they did not overlap too much. The higher we choose n, the more overlap we can tolerate but the stronger restrictions of other sorts we must build into the comprehension schema. Moreover, we will never be able to get something such that one of its pseudo-members is part of another one.

Membership$_3$

Let us put aside the task of providing for touching or overlapping pseudo-members and turn to the opposite task of providing for disconnected pseudo-members. Our plan will be to provide strings to tie together the disconnected parts of each pseudo-member.

Let us call x a *member*$_3$ of y iff x is the interior of a maximal connected part of y. For any non-empty set S of connected, solid things whose closures do not touch, there is a unique thing z whose members$_3$ are all and only the members of S: namely, the sum of the closures of the members of S. Also there is something z with no members$_3$: namely, anything with no interior. This memberless$_3$ thing is not unique, but it almost is; any two memberless$_3$ things are almost identical. For any connectible but disconnected thing x, there is something z whose sole member$_3$ is x: namely, any connection of x. It is not unique, but any two connections of x are almost identical. Finally, for any set S of connectible things whose closures do not touch (except in certain exceptional cases to be discussed), there is something z whose members$_3$ are all and only the members of S: namely, a sum of connections of the members of S chosen in such a way that no two of the connections touch. This thing z is not in general unique, but any two such things are almost identical.[2]

Trouble arises when the members of S are so crowded that there is no room for all the strings required to connect the disconnected members of S. In other words, there may be no way to choose connections of members of S so that no two of the connections touch. If so, there will be nothing whose members$_3$ are all and only the members of S.

In the worst case, suppose x and y are members of S, x is hollow, y is disconnected, part of y is inside x, and part of y is out-

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2 We might overcome this weakening of extensionality if we had a predicate *precedes* such that for anything z there was a unique thing w having exactly the same members$_3$ as z and preceding everything else having exactly those members$_3$. Then we could take w as *the* pseudo-set with those pseudo-members, taking x to be a *member*$_{3+}$ of y iff x is a member$_3$ of y and y precedes everything else having exactly those members$_3$. I do not know how to define such a precedence-relation, however, without taking some further primitives.

side x. Then no matter how we connect y, the connection of y will overlap—and hence touch—x. Or suppose x is bottle-shaped, and suppose many disconnected members of S each have a part inside x and a part outside. They will all have to be connected by strings through the neck of x and there may not be room for that many strings to pass through the neck without touching each other or touching the closure of x.

The problem of crowding will not arise if S meets the following condition: there is something y such that some connection of any member of S is part of y and such that no connected part of y overlaps more than one member of S. (This condition also implies that the closures of members of S do not touch.) Unfortunately, it is complicated and not as distant as might be desired from the mere statement that there is something whose members_3 are all and only the members of S. Perhaps some simpler sufficient condition for non-crowding can be found.

We have a weakened principle of extensionality, a restricted principle of comprehension, and an unrestricted principle of *Aussonderung:*

E3. If all and only members_3 of x are members_3 of y, then x and y are almost identical.

C3. There is something z having as members_3 all and only those things x such that __x__, provided that: (1) whenever __x__, x is connectible; and (2) there is something y such that some connection of anything x such that __x__ is part of y and such that no connected part of y overlaps more than one thing x such that __x__.

A3. There is something z having as members_3 all and only those members_3 x of y such that __x__.

Membership_3 is almost irreflexive and intransitive. The exception is again the universe, which is its own sole member_3. Nothing else is a member_3 of itself. If x is a member_3 of y and y is a member_3 of z, then x is not a member_3 of z unless x, y, and z are all the universe. Membership_3 is not a nominalistic generating relation according to Goodman's condition. Let the $\textit{content}_3$ of x be the set of things which bear the ancestral of membership_3 to x but have no members_3 different from themselves; then different things can have the same content_3. The content_3 of the universe is the unit set of the universe; the content_3 of anything else turns out to be a set of one or more of the almost identical memberless$_3$ (because interiorless) things.[3]

Among the disconnected things which may be members$_3$ are pseudo-sets with more than one member$_3$. With membership$_3$ we begin to get something like the ordinary system of cumulative types. Suppose x and y are solid and connected; and suppose they are rather small and far apart so that there is no problem of crowding. Let the result of enclosing a list in braces now denote an arbitrarily chosen one of the things having as members$_3$ all and only the listed things. (In view of the arbitrary choice, this notation cannot be part of our construction.) Then we have, for instance, the following distinct things:

$$x \qquad \{x\} \qquad \{\{x\}\} \qquad \{\{\{x\}\}\}$$
$$\{x, y\} \qquad \{\{x, y\}\} \qquad \{\{x\}, y\} \qquad \{\{x\}, \{y\}\}$$
$$x + y \qquad \{x\} + y \qquad \{x\} + \{y\} \qquad \{x + y\}$$

However, we do not have these:

$$\{x, \{x\}\} \qquad \{x, \{x, y\}\} \qquad \{x, x + y\}$$

Nor do we go on forever generating pseudo-sets of higher and higher type. We cannot: pseudo-sets are, after all, only sums of our finitely many atoms, so there can be only finitely many of them. Sooner or later crowding will set in and the proviso of C3 will fail.

If dogs and cats and were made out of our atoms, did not touch one another, were connectible, and were uncrowded, then we could handle another of Goodman's problematic sentences: "There are more cats than dogs." ([1], section II, 3. Goodman handles this sentence by means of primitive predicates pertaining to size). We would say that there is something z such that (1) every member$_3$ of z has one cat and one dog as its sole members$_3$; (2) no cat or dog is a member$_3$ of two different members$_3$ of z; (3) every dog is a member$_3$ of a member$_3$ of z; but (4) not every cat is a member$_3$ of a member$_3$ of z. It would be possible to shorten the definition, but at the expense of obscuring the analogy to standard set theory.

In a similar vein we can do some arithmetic, correctly until crowding interferes. Call x and y *separate but equal* iff the closures of x and y do not touch and there is something z such that x + y is part of z and such that every member$_3$ of z has one member$_3$ of x

[3] If we started with an infinite array of points, some things would have empty content$_3$.

and one $member_3$ of y as its sole $members_3$. Call x and y *equal* iff something is separate but equal both to x and to y. Call x an *arithmetical sum* of y and z iff there is something w such that: (1) all $members_3$ of w are $members_3$ of x; (2) w is equal to y; and (3) x-w is equal to z. Call x an *arithmetical product* of y and z iff there are something w and something u such that: (1) all and only $members_3$ of $members_3$ of w are $members_3$ of u; (2) u is equal to x; (3) w is equal to y; and (4) every $member_3$ of w is equal to z. Call x a *zero* if x has no $members_3$. Call x a *one* if x has exactly one $member_3$. Call x a *two* iff x is an arithmetical sum of a one and a one; call x a *three* iff x is an arithmetical sum of a two and a one; and so on.

$Membership_4$

$Membership_2$ allows pseudo-members to touch, but requires them to be connected. $Membership_3$ allows pseudo-members to be disconnected but requires them not to touch. How might we combine the merits of the two? Let us call x a *$member_4$* of y iff x is the closure of the closure of the interior of a maximal connected part of y.

To see how this will work, suppose we want something z whose sole $members_4$ are x and y, disconnected things which touch but do not overlap each other. Only very well-behaved things are eligible to be $members_4$; we must assume that (1) x and y have interiors r and s respectively; (2) x and y are stable; (3) r and s have interiors t and u respectively; (4) r and s are stable; (5) t and u are connectible; and (6) t and u have connections v and w respectively such that v and w do not touch. (Assumption (6) is not unreasonable in view of the fact that the closures of t and u do not touch.) Now let z be v + w. When we take the maximal connected parts of z we get v and w. When we take the interiors of v and w we get t and u. When we take the closures of t and u we get r and s. Finally, when we take the closures of r and s—that is, the $members_4$ of z—we get x and y.

The theory of $membership_4$ resembles that of $membership_3$; we need not examine it at length. The principal difference is that the standards of eligibility for pseudo-membership are raised: if x is a $member_4$ of anything, x must be stable, the interior of x must be stable, and the interior of the interior of x must be connectible.

We could also allow disconnected pseudo-members to overlap slightly. Call x a *$member_4^0$* of y iff x is a $member_4$ of y; call x a *mem-*

ber$_4^{n+1}$ of y iff x is a closure of a member$_4^n$ of y. Of course we must pay for our increasing tolerance of overlap by elevating still further our other standards of eligibility for pseudo-membership; and even so we will not get pseudo-sets such that one pseudo-member is part of another.

Membership$_5$

Except for the almost identical zeros, everything has members$_3$; and except for the universe, everything has as its content$_3$ a set of one or more zeros. (The situation for membership$_4$ is analogous.) We have fallen into Pythagoreanism: everything except the universe is made out of pseudo-Zermelo numbers. This purity might please some. Others might wish to provide for *Urelements:* things without pseudo-members which are not all almost identical.

Call x an *Urelement* iff x is stable. Call x a *member*$_5$ of y iff x is a member$_3$ of y and y is not an *Urelement.* We can easily destabilize something by adding to it another atom that does not touch the rest of it (unless it is almost identical with the universe, in which case no such atom is available). So for anything x (not almost identical with the universe) there is a non-*Urelement* z with the same members$_3$. Hence the members$_5$ of z are all and only the members$_3$ of x.

No *Urelement* or zero has members$_5$. Everything memberless$_5$ is either an *Urelement* or a zero. Zeros are not *Urelements* because they have no interiors. *Urelements* are not, in general, almost identical with one another. Some *Urelements,* however, are almost identical with other *Urelements,* with zeros, or with things having members$_5$.

The weakened principle of extensionality must of course be modified:

E5. If all and only members$_5$ of x are members$_5$ of y and if neither x nor y is an *Urelement,* then x and y are almost identical.

The anti-crowding proviso of the comprehension principle must be strengthened slightly. If connecting strings are required to form the pseudo-set of given things, they will do the destabilizing; but if not, we will need to make sure there is room for a destabilizing atom. It is sufficient to require that the thing y mentioned in the proviso of C3 not be almost identical with the universe.

The universe is no longer a pseudo-member of itself. Membership$_5$ is irreflexive and intransitive. The universe is anomalous in a new way: it is an *Urelement* too big to be a member$_5$ of anything. The same is true of any *Urelement* almost identical with the universe, and in general of any *Urelement* that is not connectible.

Defining the *content*$_5$ of x as usual, the content$_5$ of anything turns out to be a set of zeros and *Urelements.*

Membership$_6$

Membership$_6$ combines the merits of membership$_4$ and membership$_5$. Its definition, and that of membership$^n_6$ for any given n, are left to the reader.

Membership$_7$

We turn now to another pseudo-membership relation having two advantages over membership$_3$ and modifications thereof: (1) It handles disconnected pseudo-members without connecting strings, and hence without problems of crowding of the strings. We can even permit the pseudo-members of a pseudo-set to exhaust the entire universe, thus leaving no room for connecting strings. (2) It tolerates severe overlap of pseudo-members. We can even permit one pseudo-member of a pseudo-set to be a proper part of another. We buy these advantages, however, at a high price: we cannot have pseudo-sets of things that are too small, or that overlap only slightly, or that are themselves pseudo-sets.

Suppose we are given a set S of things. Members of S may be disconnected; they may together exhaust the universe; they may overlap; some may be proper parts of others. We may break the members of S into connected, non-overlapping fragments in such a way that every member of S is the sum of one or more of the fragments. Disconnected members of S will yield more than one fragment; overlapping members of S will yield fragments belonging to more than one member of S. We take interiors of the fragments, so that we may recover the fragments as closures of maximal connected parts of a pseudo-set; and we label the interiors of the fragments to carry information telling us which fragments to sum together to recover the members of S. All fragments which are part of any one member of S are to bear matching labels. Fragments which are part of two or more overlapping members of S must accordingly bear two or more labels.

In order to avoid crowding, we shall put each label of a fragment in a cavity hollowed out inside that fragment. Thus the label will be part of the interior of the fragment, made recognizable by removing some of the surrounding atoms. By making each label-holding cavity no larger than necessary, we can make sure that when we take the closure of the labeled interior of a fragment we will fill the cavities, and thereby erase the labeling and recover the entire fragment.

We may use as our labels connections of sums of closures of atoms, the atoms being sufficiently far apart so that the closures of closures of any two of them do not touch. Labels match when they are built around the same number of atoms; that is, when their interiors are equal.[4] Each label is surrounded by a skin and connected to the wall of its cavity by a string in such a way that the labeled interior of a fragment is connected, and each label is a maximal connected part of the interior of the labeled interior of its fragment.

Finally, we take as our pseudo-set corresponding to S the sum of labeled interiors of fragments. Under favorable conditions, such a thing will exist. We recover its pseudo-members according to the following definition: x is a *member*$_7$ of y iff there is something z such that (1) z is a maximal connected part of the interior of some maximal connected part of y, and (2) x is the sum of the closures of all and only those maximal connected parts w of y such that the interior of some maximal connected part v of the interior of w is equal to the interior of z. (That is, z is a label and x consists of the closures of those maximal connected parts w of y such that w bears a label v that matches z.)

We do not yet have even our usual weakened principle of extensionality, since if x and y do not touch, y has an interior, and y bears no label, then x and x + y will have the same members$_7$ but will not be almost identical. Let us call x *minimal* iff no proper part of x has the same members$_7$ as x. Then we have weakened extensionality as follows:

E7. If all and only members$_7$ of x are members$_7$ of y and if x and y are minimal, then x and y are almost identical.

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[4] Here we employ our previous definition of equality in terms of membership$_3$; and so our alternative to the direct use of connecting strings depends on a subsidiary use of them. However, it is unlikely that we will be troubled by crowding of strings. In comparing two interiors of labels there is no harm in letting a string run through regions occupied by members of S. We can ignore everything except the two things we are comparing and the other strings used in comparing them.

(Note that the only memberless$_7$ things that are minimal are the atoms, which are almost identical to each other; they are our empty pseudo-sets.) If we liked, we might revise the definition of pseudo-membership so that only minimal things have pseudo-members at all.

The principle of comprehension is restricted in a rather complicated way; we shall not attempt to state it. If there is to be a pseudo-set of the things x such that __x__, it must be possible to break those things into stable, well-connected fragments large enough to contain their labels. We will be unable to do this if some of the desired pseudo-members are too small, or if some of them overlap only slightly. The more pseudo-members there are, the larger must some of the labels be; the more overlapping pseudo-members share a common fragment, the more labels must be put inside that fragment. We are not limited by external crowding of strings, but instead by internal crowding of labels within the fragments.

We have an unrestricted principle of *Aussonderung.*

Defining the *content*$_7$ of x as usual, we have different things with the same content$_7$; but only because of the weakening of extensionality, since the content$_7$ of anything is simply the set of its members$_7$. Nothing both has and is a member$_7$. We cannot construct pseudo-sets of higher than first type using membership$_7$; to do so it would be necessary to put a label inside a label, which is impossible.[5] So although membership$_7$ serves rather well for pseudo-sets of first type, because of its tolerance of overlap, it cannot be used in most set-theoretic constructions.

The set-theoretic constructions that cannot be carried out using membership$_7$, for lack of pseudo-sets of higher than first type, can often be replaced by constructions using both membership$_7$ and membership$_8$. A pseudo-set based on membership$_7$ gives us a representing relation, defined as follows: x *represents* y *with respect to* z iff y is a member$_7$ of z and for every maximal connected part w of z, w is part of y if and only if there is a maximal connected part v of the interior of w such that the interiors of v and x are equal. (That is: x matches the labels that unite the fragments of y in z.) We can use membership$_7$ in this way to obtain a repre-

[5] It might be possible to overcome this difficulty by using some other method of labeling, probably at the cost of worsening the problem of internal crowding.

senting relation for the things we are interested in, provided there is something z having those things among its $members_7$. Then instead of constructing pseudo-sets of high type out of the $members_7$ of z themselves, we can rather construct pseudo-sets of high type out of their representatives. We can do this using $membership_3$, which is good at dealing with high types. $Membership_3$ cannot handle overlap, but that does not matter. Since anything equal to a representative of a given thing also represents that thing (with respect to z), each $member_7$ of z has many representatives scattered around the universe. In constructing pseudo-sets of high type by means of $membership_3$ out of representatives of $members_7$ of z, we are free to use many representatives of the same thing, chosen in such a way as to avoid overlap.

Suppose we want to say that among the political divisions of California at all levels—counties, cities, precincts, school districts, assembly districts, etc.—more have Democratic majorities than have Republican majorities. (We may pretend that these political divisions are made out of our atoms.) We could not say this using our earlier methods because of overlap: a Democratic school district may overlap a Democratic assembly district and be a proper part of a Democratic county. But we may say this: there exist x, y, z, such that (1) every political division of California with a Democratic or Republican majority is a $member_7$ of z, (2) every political division of California with a Democratic or Republican majority is represented, with respect to z, by exactly one $member_3$ either of x or of y, (3) every $member_3$ of x represents, with respect to z, a political division of California with a Democratic majority, (4) every $member_3$ of y represents, with respect to z, a political division of California with a Republican majority, and (5) y is equal to something w having as $members_3$ some but not all $members_3$ of x and no other things.

We can extend our representing relation to sets of arbitrarily high cumulative type constructed out of $members_7$ of something z by means of the following recursive clause: x *represents* S *with respect to* z iff for each member y of S there is exactly one $member_3$ w of x, and for each $member_3$ w of x there is some member y of S, such that w represents y with respect to z. (This recursive specification of a relation involving sets, unlike our original definition of representation of individuals, is no part of our nominalistic construction.)

For example, if x and y are $members_7$ of z, the ordered pair of x and y is represented with respect to z by anything w having exactly two $members_3$ u and v such that (1) u has exactly one $member_3$ r, (2) v has exactly two $members_3$ s and t, (3) r and s both represent x with respect to z, and (4) t represents y with respect to z. Note that the two representatives r and s of x must be different.

For another example, x represents with respect to z the set of all sets of $members_7$ of z iff: (1) each $member_3$ of a $member_3$ of x represents a $member_7$ of z with respect to z, and (2) for anything y such that each $member_3$ of y represents a $member_7$ of z with respect to z, there is a $member_3$ w of x such that the $members_3$ of w and the $members_3$ of y represent, with respect to z, exactly the same $members_7$ of z.

In this way, until external crowding among representatives interferes, we can represent any set, no matter how high its type, whose content consists of $members_7$ of something z.

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