Analog and Digital

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The distinction between analog and digital representation of numbers is well understood in practice. Yet its analysis has proved troublesome. I shall first consider the account given by Nelson Goodman and offer examples to show that some cases of analog representation are mis-classified, on Goodman's account, as digital. Then I shall offer alternative analyses of analog and digital representation.

1. Differentiated Analog Representation

According to Goodman in *Languages of Art*,[1] the distinction between digital and analog representation of numbers is as follows. Digital representation is *differentiated*. Given a number-representing "mark"—an inscription, vocal utterance, pointer position, electrical pulse, or whatever—it is theoretically possible, despite our inability to make infinitely precise measurements, to determine exactly which other marks are copies of the given mark and to determine exactly which number (or numbers) the given mark and its copies represent. Analog representation, on the other hand, fails to be differentiated because it is *dense*. For any two marks that are not copies, no matter how nearly indistinguishable they are, there could be a mark intermediate between them which is a copy of neither; and for any two marks that are not copies and represent different numbers, no matter how close the numbers are, there is an

[1] (Indianapolis and New York: Bobbs-Merrill, 1968), sections IV.2, IV.5, and IV.8. I have combined Goodman's syntactic and semantic differentiation and combined his syntactic and semantic density; and I have not defined differentiation and density in full generality, but only as applied to the representation of numbers.

intermediate number which would be represented by a mark that is a copy of neither.

It is true and important that digital representation is differentiated, and that it differs thereby from the many cases of analog representation that are undifferentiated and dense: those cases in which all real numbers in some range are represented by values of some continuously variable physical magnitude such as voltage. But there are other cases: representation that is as differentiated and non-dense as any digital representation and yet is analog rather than digital representation. Here are two examples of differentiated analog representation. If accepted, they show that Goodman's distinction, interesting though it is in its own right, does not coincide with the analog-digital distinction of ordinary technological language.

Example 1: ordinary electrical analog computers sometimes receive their numerical inputs in the form of settings of variable resistors. A setting of 137 ohms represents the number 137, and so on. There are two ways these variable resistors might work. In the first case, a contact slides smoothly along a wire with constant resistance per unit length. In the second case, there is a switch with a very large but finite number of positions, and at each position a certain number of, say, 1-ohm fixed resistors are in the circuit and the rest are bypassed. I do not know which sort of variable resistor is used in practice. In either case, the computer is an analog computer and its representation of numbers by electrical resistances is analog representation. In the sliding-contact case the representation is undifferentiated and dense; but in the multi-position switch case the representation is differentiated and non-dense, yet analog and not digital.

Example 2: we might build a device which works in the following way to multiply two numbers x and y. There are four receptacles: W, X, Y, and Z. We put a large amount of some sort of fluid —liquid, powder, or little pellets—in W. We put x grams of fluid in X, y grams in Y, and none in Z. At the bottom of X is a valve, allowing fluid to drain at a constant rate from X into a wastebasket. At the bottom of W is a spring-loaded valve, allowing fluid to drain from W into Z at a rate proportional to the amount of fluid in Y. For instance, if Y contains 17 grams of fluid, then the rate of drainage from W is 17 times the constant rate of drainage from X. We simultaneously open the valves on W and X; as soon as X is empty, we close the valve on W. The amount z of fluid that has passed from W to Z at a rate proportional to y in a time proportional to x is the product of the numbers x and y. This device is an analog computer, and its representation of numbers by amounts of fluid is analog representation. In case the fluid is a liquid, the representation is undifferentiated and dense (almost—but even a liquid consists of molecules); but in case it is 1-gram round metal pellets, the representation is differentiated and non-dense, yet analog and not digital.

2. Analog Representation

It is commonplace to say that analog representation is representation of numbers by physical magnitudes. And so it is; but so is digital representation, or any other sort of representation that could be used in any physically realized computer, unless we adopt a peculiarly narrow conception of physical magnitudes.

We may regard a physical magnitude as a function which assigns numbers to physical systems at times. A physical magnitude may be defined for every physical system at every moment of its existence, or it may be defined only for physical systems of some particular kind or only at some times. Let us call any such function a *magnitude*. We can then define a *physical magnitude* as any magnitude definable in the language of physics more or less as we know it. This is imprecise, but properly so: the vagueness of "physical magnitude" ought to correspond to the vagueness of "physics".

Since the language of physics includes a rich arithmetical vocabulary, it follows that if the values of a magnitude depend by ordinary arithmetical operations on the values of physical magnitudes, then that magnitude is itself a physical magnitude. Now suppose that in computers of a certain sort, numbers are represented in the following way. We consider the voltages $v_0, \ldots, v_{35}$ between 36 specified pairs of wires in any such computer at any time at which the computer is operating. By taking these voltages in order, and associating with each positive voltage the digit 1 and with each negative voltage the digit 0, we obtain a binary numeral and the number denoted by that numeral. Here is as good a case of digital representation as we could find. Yet it is also a case of representation by a physical magnitude: the number represented thus by computer s at time t is the value of the physical magnitude defined below by means of arithmetical vocabulary in terms of the voltages $v_0, \ldots, v_{35}$. (We take the magnitude to be undefined if any v_i is 0.)

$$V(s,t) = \sum_{i=0}^{35} 2^i \begin{Bmatrix} 1 \text{ if } v_i(s,t) > 0 \\ 0 \text{ if } v_i(s,t) < 0 \end{Bmatrix}$$

$$= \sum_{i=0}^{35} 2^i \left[\frac{1}{2} + \frac{\sqrt{v_i(s,t)^2}}{2v_i(s,t)} \right]$$

Analog representation, then, is representation of numbers by physical magnitudes of a special kind. Resistances, voltages, amounts of fluid, for instance, are physical magnitudes of the proper kind for analog representation; but V, as defined above, is a physical magnitude not of the proper kind for analog representation. How may we distinguish physical magnitudes that are of the proper kind for analog representation from those that are not?

We might try saying that the magnitudes suitable for analog representation are those that are expressed by primitive terms in the language of physics. This will not do as it stands: a term is primitive not relative to a language but relative to some chosen definitional reconstruction thereof. Any physical magnitude could be expressed by a primitive term in a reconstruction designed *ad hoc*, and there is no physical magnitude that must be expressed by a primitive term. We may, however, define a *primitive magnitude* as any physical magnitude that is expressed by a primitive term in some *good* reconstruction of the language of physics—good according to our ordinary standards of economy, elegance, convenience, familiarity. This definition is scarcely precise, but further precision calls for a better general understanding of our standards of goodness for definitional reconstructions, not for more work on the topic at hand.

Even taking the definition of primitive magnitudes as understood, however, it is not quite right to say that analog representation is representation by primitive magnitudes. Sometimes it is representation by physical magnitudes that are *almost primitive*: definable in some simple way, with little use of arithmetical operations, in terms of one or a few primitive magnitudes. (Further precision here awaits a better general understanding of simplicity of definitions.) Products of a current and a voltage between the same two points in a circuit, or logarithmically scaled luminosities, seem unlikely to be expressed by primitive terms in any good reconstruction. But they are almost primitive, and representation of numbers by them would be analog representation. Of more interest for our present purposes, rounded-off primitive magnitudes are almost primitive. Such are the number-representing magnitudes in our two examples of differentiated analog representation: resistance rounded to the nearest ohm and amount of fluid rounded to the nearest gram. We use the rounded-off magnitudes because, in practice, our resistors or pellets are not exactly one ohm or gram each; so if we take the resistance of 17 resistors or the mass of 17 pellets to represent the number 17 (rather than some unknown number close to 17) we are using not primitive magnitudes but almost primitive magnitudes.

The commonplace definition of analog representation as representation by physical magnitudes is correct, so far as I can see, if taken as follows: *analog representation* of numbers is representation of numbers by physical magnitudes that are either primitive or almost primitive according to the definitions above.

3. Digital Representation

It remains to analyze digital representation, for not all non-analog representation of numbers is digital. (It may be, however, that all practically useful representation of numbers is either analog or digital.) If numbers were represented by the physical magnitude defined as follows in terms of voltages between specified pairs of wires in some circuit, the representation would be neither analog nor digital (nor useful).

$$P(s,t) = \sum_{i=0}^{35} \log_{10}(\sinh(\sqrt{v_i(s,t)}))$$

Digital representation is representation by physical magnitudes of another special kind, the kind exemplified above by V.

We may first define an *n-valued unidigital magnitude* as a physical magnitude having as values the numbers $0, 1, \ldots, n-1$ whose values depend by a step function on the values of some primitive magnitude. Let U be an n-valued unidigital magnitude; then there is a primitive magnitude B which we may call the *basis* of U and there is an increasing sequence of numbers $a_1, \ldots, a_{n-1}$ which we may call the *transition points* of U and, for each system s on which U is defined, there is a part p of s such that U is defined as follows on s.

$$U(s,t) = \begin{cases} 0 \text{ if } a_1 > B(p,t) \\ 1 \text{ if } a_2 > B(p,t) > a_1 \\ \cdot \\ \cdot \\ \cdot \\ n - 1 \text{ if } B(p,t) > a_{n-1} \end{cases}$$

$$= \sum_{i=1}^{n-1} \left[\frac{1}{2} + \frac{\sqrt{(B(p,t) - a_i)^2}}{2(B(p,t) - a_i)} \right]$$

Let us call a unidigital magnitude *differentiated* if, in the systems on which it is defined, its basis does not take values at or near its transition points.

Representation of numbers by differentiated unidigital magnitudes, or by physical magnitudes whose values depend arithmetically on the values of one or more differentiated unidigital magnitudes, is differentiated and non-dense representation, hence digital representation in Goodman's sense. But it may not really be digital representation. In fact, it may be analog representation. A unidigital magnitude with many evenly spaced transition points is exactly what we have been calling a rounded-off primitive magnitude; it is almost primitive and suitable for analog representation. The number-representing magnitudes in our examples of differentiated analog representation are differentiated unidigital magnitudes, but representation by these magnitudes is analog representation.

What distinguishes digital representation, properly so-called, is not merely the use of differentiated unidigital magnitudes; it is the use of the many combinations of values of a few few-valued unidigital magnitudes. Let us now define a *multidigital magnitude* as any physical magnitude whose values depend arithmetically on the values of several few-valued unidigital magnitudes. Let us call a multidigital magnitude *differentiated* if it depends on differentiated unidigital magnitudes. In fixed point digital representation, for instance, a multidigital magnitude M depends as follows on several n-valued unidigital magnitudes $u_0, \ldots, u_{m-1}$.

$$M(s,t) = \sum_{i=0}^{m-1} n^i u_i(s,t)$$

V is a multidigital magnitude of this sort, with $m = 36$, $n = 2$, and each u_i having as its basis the corresponding voltage v_i at the specified part of the system. Other multidigital magnitudes depend in more complicated ways upon their unidigital magnitudes. Most often, the unidigital magnitudes are not merely few-valued but two-valued; but not so, for instance, in an odometer in which the unidigital magnitudes are ten-valued step functions of angles of rotation of the wheels.

I suggest, therefore, that we can define *digital representation* of numbers as representation of numbers by differentiated multidigital magnitudes.